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Vilém Vychodil

A note on congruence permutability and fuzzy logic

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Abstract We study congruence permutability of algebras with fuzzy equalities. The notion of degree of congruence permutability naturally arises in this context. We present a Mal'cev-like characterization of congruence permutable varieties of algebras with fuzzy equalities. Our note presents a way to generalize various congruence conditions from the point of view of fuzzy logic.

Keywords Fuzzy equality · Fuzzy logic · Mal'cev-condition · Permutability

1 Introduction

The notion of a congruence permutability has been the subject of profound study in universal algebra and related mathematical disciplines. The fundamental contribution is due to Mal'cev and his characterization [14] of permutable varieties. The key issue of [14] is that an algebraic property (permutability of a variety) is equivalently expressed using logical terms – by the existence of a ternary term p satisfying certain identities involving p. Later on, the effort in studying related properties of varieties resulted in characterizing so-called Mal'cev conditions [15, 18].

Universal algebra is closely connected with the equational fragment of first-order logic. From the viewpoint of first-order logic, algebras can be thought of as first-order structures for languages with function symbols and a single relation symbol, the symbol of equality. The relational part of an algebra is trivial because the equality predicate is interpreted in each algebra by the identity relation ($\mathfrak{u}$ equals $\mathfrak{v}$, written $\mathfrak{u} \approx \mathfrak{v}$, iff $\mathfrak{u}$ and $\mathfrak{v}$ are identical elements). The present paper deals with congruence permutability of algebras extended by a nontrivial relational part. The extended algebras, so-called algebras with fuzzy equalities, naturally appear if we develop the theory of universal algebras in the context of predicate fuzzy logics with function symbols.

Fuzzy logic in narrow sense [1, 11, 12, 16] provides us with a formal framework suitable to study structures equipped with so-called fuzzy relations, i.e. relations, where elements are related to certain degree. In our previous results [2–6], we have presented *algebras with fuzzy equalities* as the semantic structures for the equational fragment of predicate fuzzy logic with truth-evaluated syntax. It is worth to add that the concept in question is by no means artificial. An algebra with fuzzy equality is a set with operations on it that is equipped with *similarity* $\approx$ (a particular fuzzy equivalence relation) such that each operation f is compatible with $\approx$. The compatibility ensures that each f yields similar results if applied to pairwise similar arguments.

Note in advance that the functional part of an algebra with fuzzy equality is a classical algebra, i.e. a set M equipped with mappings $f: M^n \to M$. During the past three decades there appeared several approaches to extending classical algebras in fuzzy setting. The investigation of algebras with "vague universes" started with Rosenfeld's fuzzy groups [17]. Another approach based on fuzzy functions [8] can be seen as a treatment of algebras with "vague operations". Contrary to that, our approach is primarily not intended to be another generalization of the notion of an algebra. We are basically interested in properties of model classes of equational theories rather than properties of particular fuzzy structures. Working with more general structures (e.g. structures with fuzzy functions) might of course lead to more general results but we will purposely not go into this.

In what follows, we present a Mal'cev-like characterization of congruence permutability in graded style. Section 2 summarizes basic notions and presents the main result. In Sect. 3, we elaborate the proof and give some remarks.

2 Congruence permutability in fuzzy setting

We use a *complete residuated lattice* as the structure of truth degrees. Complete residuated lattices were introduced into

V. Vychodil
Department Computer Science,
Palacký University,
Tomkova 40, CZ-779 00,
Olomouc, Czech Republic
E-mail: vilem.vychodil@upol.cz

the context of fuzzy logic by Goguen [9,10]. Particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras, see [1, 12,13,16]. This includes, among other structures, residuated lattices given by continuous t-norms [1,12].

A (*complete*) *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, (2) $\langle L, \otimes, 1\rangle$ is a commutative monoid, (3) $\langle \otimes, \rightarrow\rangle$ is an *adjoint pair,* i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for each $a, b, c \in L$ (*adjointness property*). For brevity, let $(a \rightarrow b) \wedge (b \rightarrow a)$ be denoted by $a \leftrightarrow b$. In the sequel, $\mathbf{L}$ always denotes a complete residuated lattice.

An $\mathbf{L}$-*set* A (or *fuzzy set with truth degrees in* $\mathbf{L}$) in a *universe set* U is any mapping $A : U \rightarrow L$, $A(\mathfrak{u}) \in L$ being interpreted as the truth degree of "element $\mathfrak{u}$ belongs to A". For every $\mathbf{L}$-set $A\colon U \rightarrow L$, we define a classical set $\mathrm{Supp}(A)$ by $\mathrm{Supp}(A) = \{\mathfrak{u} \in U;\ A(\mathfrak{u}) > 0\}$. $\mathrm{Supp}(A)$ is called the *support set of* A. For an $\mathbf{L}$-set $A : U \rightarrow L$ and $a \in L$, we define the *a-cut* ${}^{a}/A$ *of* A by putting ${}^{a}/A = \{\mathfrak{u} \in U;\ A(\mathfrak{u}) \geq a\}$. For $\mathbf{L}$-sets A and B in U we define degrees $S(A, B), E(A, B) \in L$ as follows:

$$S(A, B) = \textstyle\bigwedge_{\mathfrak{u}\in U}\big(A(\mathfrak{u}) \rightarrow B(\mathfrak{u})\big), \tag{1}$$

$$E(A, B) = \textstyle\bigwedge_{\mathfrak{u}\in U}\big(A(\mathfrak{u}) \leftrightarrow B(\mathfrak{u})\big). \tag{2}$$

$S(A, B)$ is called the *subsethood degree of A in B,* $E(A, B)$ is called the *equality degree of A and B*. We have $E(A, B) = S(A, B) \wedge S(B, A)$. Furthermore, we write $A \subseteq B$ (A is a *subset* of B) iff $S(A, B) = 1$, i.e. iff $A(\mathfrak{u}) \leq B(\mathfrak{u})$ for each $\mathfrak{u} \in U$. The operations with $\mathbf{L}$-sets are defined componentwise, i.e. for $\mathbf{L}$-sets A, B in U we consider $A \cap B\colon U \rightarrow L$ such that $(A \cap B)(\mathfrak{u}) = A(\mathfrak{u}) \wedge B(\mathfrak{u})$ for each $\mathfrak{u} \in U$, analogously for $\cup$. A *binary* $\mathbf{L}$-*relation* R (or *binary fuzzy relation with truth degrees in* $\mathbf{L}$) on a *universe set* U is an $\mathbf{L}$-set in the universe set $U \times U$, i.e. it is a mapping $R : U \times U \rightarrow L$. An $\mathbf{L}$-*equivalence* (or *similarity*) relation E on a set U is a mapping $E : U \times U \rightarrow L$ satisfying the following conditions: $E(\mathfrak{u}, \mathfrak{u}) = 1$ (reflexivity), $E(\mathfrak{u}, \mathfrak{v}) = E(\mathfrak{v}, \mathfrak{u})$ (symmetry), $E(\mathfrak{u}, \mathfrak{v}) \otimes E(\mathfrak{v}, \mathfrak{w}) \leq E(\mathfrak{u}, \mathfrak{w})$ (transitivity) for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in U$. An $\mathbf{L}$-equivalence on U for which $E(\mathfrak{u}, \mathfrak{v}) = 1$ iff $\mathfrak{u} = \mathfrak{v}$ will be called an $\mathbf{L}$-*equality.* Function $f : U^n \rightarrow U$ is said to be *compatible* with binary $\mathbf{L}$-relation R on U if

$$\begin{aligned} &R(\mathfrak{a}_1, \mathfrak{b}_1) \otimes \cdots \otimes R(\mathfrak{a}_n, \mathfrak{b}_n)\\ &\quad\leq R\big(f(\mathfrak{a}_1, \ldots, \mathfrak{a}_n), f(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)\big), \end{aligned} \tag{3}$$

for all $\mathfrak{a}_1, \mathfrak{b}_1 \ldots \mathfrak{a}_n, \mathfrak{b}_n \in U$.

A *type,* denoted by F, is a set of function symbols $f \in F$ together with their arities (since we mostly work with a fixed type, we will not mention it explicitly). An *algebra with* $\mathbf{L}$-*equality* (of type F) is a triplet $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$, where $\langle M, F^{\mathbf{M}}\rangle$ is a classical algebra [7] of type F and $\approx^{\mathbf{M}}$ is an $\mathbf{L}$-equality on M such that each function $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$. An algebra with $\mathbf{L}$-equality will also be simply called an $\mathbf{L}$-algebra. In [4] we developed algebraic constructions (morphisms, subalgebras, direct products) respecting both the functional and the (nontrivial) relational part of $\mathbf{L}$-algebras. Classes of $\mathbf{L}$-algebras will be denoted $\mathcal{K}, \mathcal{K}', \ldots$ A class $\mathcal{K}$ of $\mathbf{L}$-algebras closed under homomorphic images, subalgebras, and direct products is called a *variety of* $\mathbf{L}$-*algebras.* There is a characterization of varieties of $\mathbf{L}$-algebras [3], which is an analog to the well-known Birkhoff's variety theorem (varieties of $\mathbf{L}$-algebras are exactly the model classes of *fuzzy sets of identities*). In the sequel, we briefly summarize the constructions involved in the subsequent characterization. In particular, we focus on properties which are trivial or degenerated in the classical case. For details we refer the reader to [1,4].

An $\mathbf{L}$-relation θ on M is called a *congruence on* $\mathbf{M}$ if (i) θ is an $\mathbf{L}$-equivalence relation on M, (ii) $\approx^{\mathbf{M}} \subseteq \theta$, (iii) all functions $f^{\mathbf{M}} \in F^{\mathbf{M}}$ are compatible with θ. Let $\mathrm{Con}(\mathbf{M})$ denote the set of all congruences on $\mathbf{M}$. It can be shown [4] that $\mathrm{Con}(\mathbf{M})$ is closed under arbitrary intersections. For a binary $\mathbf{L}$-relation R on M we denote by $\theta(R)$ the least congruence on $\mathbf{M}$ containing R. Moreover, for every $\mathfrak{u}, \mathfrak{v} \in M$ and $a \in L$ we define the *principal congruence* $\theta({}^{a}/\langle\mathfrak{u}, \mathfrak{v}\rangle) \in \mathrm{Con}(\mathbf{M})$ by putting $\theta({}^{a}/\langle\mathfrak{u}, \mathfrak{v}\rangle) = \theta(R)$, where $R(\mathfrak{u}, \mathfrak{v}) = a$ and $\mathrm{Supp}(R) \subseteq \{\langle\mathfrak{u}, \mathfrak{v}\rangle\}$. Described verbally, $\theta({}^{a}/\langle\mathfrak{u}, \mathfrak{v}\rangle)$ is the congruence generated by the couple $\langle\mathfrak{u}, \mathfrak{v}\rangle$ of elements of M in the truth degree $a \in L$. The structure of principal congruences is investigated in [4].

In fuzzy approach, there are several sound definitions of a composition of fuzzy relations, which all coincide in the classical case (i.e. for $\mathbf{L}$ being the two-element Boolean algebra), see [1,11]. For our purposes, we present another generalization of one of the most common compositions of $\mathbf{L}$-relations. For each $a \in L$ and $n \in \mathbb{N}_0$ we define $a^n \in L$ by $a^0 = 1$, and $a^n = a \otimes a^{n-1}$. For binary $\mathbf{L}$-relations R_1, R_2 on U, we define binary $\mathbf{L}$-relations $R_1 \circ^n R_2$, $R_1 \circ^\omega R_2$ on U by

$$(R_1 \circ^n R_2)(\mathfrak{a}, \mathfrak{b}) = \textstyle\bigvee_{\mathfrak{c}\in U}(R_1(\mathfrak{a}, \mathfrak{c}) \otimes R_2(\mathfrak{c}, \mathfrak{b}))^n, \tag{4}$$

$$(R_1 \circ^\omega R_2)(\mathfrak{a}, \mathfrak{b}) = \textstyle\bigvee_{\mathfrak{c}\in U}\bigwedge_{n\in\mathbb{N}_0}(R_1(\mathfrak{a}, \mathfrak{c}) \otimes R_2(\mathfrak{c}, \mathfrak{b}))^n, \tag{5}$$

where $n \in \mathbb{N}_0$ and $\mathfrak{a}, \mathfrak{b} \in U$. For $n = 1$ we write $R_1 \circ R_2$ instead of $R_1 \circ^1 R_2$. Observe that in general we can have $(R_1 \circ R_2)(\mathfrak{a}, \mathfrak{b}) = 1$, although there is no $\mathfrak{c} \in U$ such that $R_1(\mathfrak{a}, \mathfrak{c}) = 1$ and $R_2(\mathfrak{c}, \mathfrak{b}) = 1$. On the other hand, such a situation cannot happen if 1 is an $\vee$-irreducible element of L.

Now we present the notion of a *graded permutability.* Recall that in the classical case, congruences θ, ϕ are called permutable iff $\theta \circ \phi = \phi \circ \theta$, which is equivalent to $\theta \circ \phi \subseteq \phi \circ \theta$. We are going to generalize the classical permutability using the subsethood degree "S", see (1), which generalizes the classical inclusion "$\subseteq$". For $\theta, \phi \in \mathrm{Con}(\mathbf{M})$, we define the degree $\mathrm{Per}(\theta, \phi)$ to which θ, ϕ are permutable by

$$\mathrm{Per}(\theta, \phi) = S(\theta \circ^\omega \phi, \phi \circ \theta).$$

For an $\mathbf{L}$-algebra $\mathbf{M}$ we put

$$\mathrm{Per}(\mathbf{M}) = \textstyle\bigwedge_{\theta,\phi\in\mathrm{Con}(\mathbf{M})}\mathrm{Per}(\theta, \phi).$$

For a variety $\mathcal{K}$ of $\mathbf{L}$-algebras we define the degree $\mathrm{Per}(\mathcal{K})$ by

$$\mathrm{Per}(\mathcal{K}) = \textstyle\bigwedge_{\mathbf{M}\in\mathcal{K}}\mathrm{Per}(\mathbf{M}). \tag{6}$$

Thus, $\mathrm{Per}(\mathcal{K})$ represents the degree to which a variety $\mathcal{K}$ is permutable. Observe that for $\mathbf{L}$ being two-element Boolean algebra, $\mathrm{Per}(\mathcal{K}) = 1$ iff $\mathcal{K}$ is permutable in the classical sense. The relationship between the classical permutability and the degree given by $\mathrm{Per}(\mathcal{K})$ will be commented later on.

Now we turn our attention to the term characterization of $\mathrm{Per}(\mathcal{K})$. First, note that *terms* and *identities* are defined as usual; $p(x_1, \ldots, x_n)$ denotes the fact that variables occurring in the term p are among $x_1, \ldots, x_n$. Let $T(X)$ denote the set of all terms in variables X. The interpretation $||p||_{\mathbf{M},v}$ of $p \in T(X)$ in $\mathbf{M}$ under a valuation $v\colon X \to M$ is defined as in the classical case. For an $\mathbf{L}$-algebra $\mathbf{M}$ and a valuation $v\colon X \to M$ we define the degree $||p \approx q||_{\mathbf{M},v} \in L$ to which an identity $p \approx q$ is true in $\mathbf{M}$ under v by $||p \approx q||_{\mathbf{M},v} = ||p||_{\mathbf{M},v} \approx^{\mathbf{M}} ||q||_{\mathbf{M},v}$. For a class $\mathcal{K}$ of $\mathbf{L}$-algebras we define the degree $||p \approx q||_{\mathcal{K}} \in L$ to which $p \approx q$ is valid in $\mathcal{K}$ by

$$||p \approx q||_{\mathcal{K}} = \bigwedge_{\mathbf{M}\in\mathcal{K}} \bigwedge_{v:X\to M} ||p \approx q||_{\mathbf{M},v}. \quad (7)$$

Recall that the Mal'cev characterization [14] of permutable varieties states that congruences of any $\mathbf{M} \in \mathcal{K}$ are permutable iff there exists a ternary term $p(x, y, z)$ such that the identities $p(x, y, y) \approx x$ and $p(x, x, y) \approx y$ are valid in $\mathcal{K}$. In fuzzy case, however, we deal with the degree of permutability $\mathrm{Per}(\mathcal{K}) \in L$ and with the truth degree $||\cdots||_{\mathcal{K}} \in L$. The following assertion presents a connection between the graded permutability and the graded validity of Mal'cev identities.

Theorem (characterization of graded permutability). *Let $\mathcal{K}$ be a variety of $\mathbf{L}$-algebras and let $X = \{x, y, z\}$. Then*

$$\mathrm{Per}(\mathcal{K}) = \bigvee_{p\in T(X)} \big(||p(x, y, y) \approx x||_{\mathcal{K}} \otimes ||p(x, x, y) \approx y||_{\mathcal{K}}\big). \quad (8)$$

That is, the degree of permutability of a variety $\mathcal{K}$ is equal to the supremum (of truth degrees connected in a conjunctive manner) which ranges over all ternary terms. It is immediate that for $\mathbf{L}$ being the two-element Boolean algebra, we obtain exactly the well-known Mal'cev characterization of congruence permutability for varieties of algebras.

The proof is presented in Sect. 3.

3 Proof and some remarks

Let p be a term in variables X. For $x \in X$ let $|p|_x$ denote the *number of occurrences of x in p*. If $T(X) \neq \emptyset$, then the *term $\mathbf{L}$-algebra over X,* written $\mathbf{T}(X)$, has the universe $T(X)$; functions are defined as usual and $p \approx^{\mathbf{T}(X)} p = 1$, $p \approx^{\mathbf{T}(X)} q = 0$ for all $p, q \in T(X)$, $p \neq q$. For a term $p(x_1, \ldots, x_n)$ and $\mathbf{M}$ we define the *term function* $p^{\mathbf{M}}\colon M^n \to M$ by $p^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = ||p||_{\mathbf{M},v}$, where $v(x_i) = \mathfrak{a}_i$ for $i = 1, \ldots, n$. Let us note that term functions (in fuzzy case) have the following interesting property which is degenerated in the classical case. Let $p(x_1, \ldots, x_n)$ be a term, $\mathbf{M}$ be an $\mathbf{L}$-algebra. Then

$$\theta(\mathfrak{a}_1, \mathfrak{b}_1)^{|p|_{x_1}} \otimes \cdots \otimes \theta(\mathfrak{a}_n, \mathfrak{b}_n)^{|p|_{x_n}} \leq \theta\big(p^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n), p^{\mathbf{M}}(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)\big), \quad (9)$$

for every $\theta \in \mathrm{Con}(\mathbf{M})$ and arbitrary $\mathfrak{a}_1, \mathfrak{b}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}_n \in M$, see [1,4]. Note that (9) can be thought of as an *estimation of the truth degree* to which the *results of the term function* $p^{\mathbf{M}}$ are *similar.* If $\otimes$ is $\wedge$, the powers $|p|_{x_i}$ can be removed from (9). This has the following consequence. In general, the estimation describing sensitivity of a term function depends on the structure of the corresponding term, hence the number of occurrences of variables in that term is important. If $\otimes = \wedge$, complexity of terms does not play any role.

For $\theta \in \mathrm{Con}(\mathbf{M})$, an $\mathbf{L}$-algebra $\mathbf{M}/\theta = \langle M/\theta, \approx^{\mathbf{M}/\theta}, F^{\mathbf{M}/\theta}\rangle$ is called the *factor $\mathbf{L}$-algebra of $\mathbf{M}$ modulo θ*, if (i) $\langle M/\theta, F^{\mathbf{M}/\theta}\rangle$ is the classical factor algebra of $\langle M, F^{\mathbf{M}}\rangle$ modulo $\{\langle \mathfrak{a}, \mathfrak{b}\rangle\,;\ \theta(\mathfrak{a}, \mathfrak{b}) = 1\}$, (ii) $[\mathfrak{a}]_\theta \approx^{\mathbf{M}/\theta} [\mathfrak{b}]_\theta = \theta(\mathfrak{a}, \mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in M$. For a variety $\mathcal{K}$ and a set X of variables, the $\mathcal{K}$-free $\mathbf{L}$-algebra $\mathbf{F}_{\mathcal{K}}(\overline{X})$ is the factor $\mathbf{L}$-algebra $\mathbf{T}(X)/\theta_{\mathcal{K}}(X)$, where $\theta_{\mathcal{K}}(X) \in \mathrm{Con}(\mathbf{T}(X))$ is defined as the intersection of a family $\{\theta \in \mathrm{Con}(\mathbf{T}(X));\ \mathbf{T}(X)/\theta \in \mathcal{K}\}$. $\mathcal{K}$-free $\mathbf{L}$-algebras behave analogously as the classical $\mathcal{K}$-free algebras, e.g. $\theta_{\mathcal{K}}(X)(p, q) = ||p \approx q||_{\mathcal{K}}$, see [3,4]. For brevity, we denote classes $[p]_{\theta_{\mathcal{K}}(X)}$ by $\overline{p}$. Furthermore, we put $\overline{X} = \{\overline{x};\ x \in X\}$. Let $\mathbf{M}, \mathbf{N}$ be $\mathbf{L}$-algebras of the same type. A mapping $h\colon M \to N$ satisfying $\mathfrak{a} \approx^{\mathbf{M}} \mathfrak{b} \leq h(\mathfrak{a}) \approx^{\mathbf{N}} h(\mathfrak{b})$ is called an *$\approx$-morphism.* Thus, an $\approx$-morphism is a mapping preserving the similarity. An $\approx$-morphism $h\colon M \to N$ is called a *morphism* (*of* $\mathbf{L}$-*algebras*) if h is a morphism between classical algebras $\langle M, F^{\mathbf{M}}\rangle$ and $\langle N, F^{\mathbf{N}}\rangle$. For a morphism $h\colon \mathbf{M} \to \mathbf{N}$ we define $\theta_h \in \mathrm{Con}(\mathbf{M})$ by $\theta_h(\mathfrak{a}, \mathfrak{b}) = h(\mathfrak{a}) \approx^{\mathbf{N}} h(\mathfrak{b})$, see [4]. Finally, for every variety $\mathcal{K}$ and $\mathbf{M} \in \mathcal{K}$, any $\approx$-morphism $h\colon \overline{X} \to \mathbf{M}$ has a uniquely determined homomorphic extension $h^\sharp\colon \mathbf{F}_{\mathcal{K}}(\overline{X}) \to \mathbf{M}$, see [4].

Lemma 1 *Let $\mathcal{K}$ be a variety of $\mathbf{L}$-algebras and let $X = \{x_1, \ldots, x_n\}$ be a set of variables. Then for every $\mathbf{M} \in \mathcal{K}$ and $\theta \in \mathrm{Con}(\mathbf{M})$ we have*

$$\theta_{\mathcal{K}}(X)\big(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n)\big) \leq \theta\big(p^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n), q^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)\big), \quad (10)$$

where $p, q \in T(X)$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \in M$.

Proof Let $v : X \to M$ be a valuation, where $v(x_i) = \mathfrak{a}_i$ for $i = 1, \ldots, n$. In much the same way as in the classical case, v induces a valuation $w : X \to M/\theta$ such that $w(x_i) = [v(x_i)]_\theta = [\mathfrak{a}_i]_\theta$ for all $i = 1, \ldots, n$. Thus,

$$\begin{aligned}
&\theta_{\mathcal{K}}(X)\big(p(x_1, \ldots, x_n), q(x_1, \ldots, x_n)\big)\\
&\quad = ||p \approx q||_{\mathcal{K}} \leq ||p \approx q||_{\mathbf{M}/\theta,w}\\
&\quad = ||p||_{\mathbf{M}/\theta,w} \approx^{\mathbf{M}/\theta} ||q||_{\mathbf{M}/\theta,w}\\
&\quad = \big[||p||_{\mathbf{M},v}\big]_\theta \approx^{\mathbf{M}/\theta} \big[||q||_{\mathbf{M},v}\big]_\theta\\
&\quad = \theta\big(||p||_{\mathbf{M},v}, ||q||_{\mathbf{M},v}\big)\\
&\quad = \theta\big(p^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n), q^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)\big),
\end{aligned}$$

which is the desired inequality. □

Lemma 2 *Let $\mathcal{K}$ be a variety of $\mathbf{L}$-algebras, $X = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ and $Y = \{x_1, \ldots, x_m, y\}$ be sets of variables.*

Then for every binary **L***-relation* R *on* $T(X)/\theta_{\mathcal{K}}(X)$ *such that* $\mathrm{Supp}(R) = \{\langle \overline{y_i}, \overline{y_j}\rangle;\ 1 \le i, j \le n\}$ *we have*

$$\begin{aligned}&\theta(R)\big(p^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y_1}, \ldots, \overline{y_n}),\\ &q^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y_1}, \ldots, \overline{y_n})\big)\\ &\quad\le \|p(x_1, \ldots, x_m, y, \ldots, y)\\ &\quad\approx q(x_1, \ldots, x_m, y, \ldots, y)\|_{\mathcal{K}}, \qquad (11)\end{aligned}$$

for arbitrary terms $p, q \in T(X)$.

Proof Take a mapping $h\colon \overline{X} \to T(Y)/\theta_{\mathcal{K}}(Y)$, where $h(\overline{x_i}) = \overline{x_i}$ $(i = 1, \ldots, m)$, and $h(\overline{y_j}) = \overline{y}$ $(j = 1, \ldots, n)$. Without loss of generality, we can assume the variable y to be among $y_1, \ldots, y_n$. For every $z_1, z_2 \in X$ let $z_{1'}, z_{2'} \in Y$ with $h(\overline{z_1}) = \overline{z_{1'}}$ and $h(\overline{z_2}) = \overline{z_{2'}}$. Now the properties of $\mathcal{K}$-free **L**-algebras yield

$$\begin{aligned}\overline{z_1} \approx^{\mathbf{F}_{\mathcal{K}}(\overline{X})} \overline{z_2} &= ||z_1 \approx z_2||_{\mathcal{K}} \le ||z_{1'} \approx z_{2'}||_{\mathcal{K}}\\ &= h(\overline{z_1}) \approx^{\mathbf{F}_{\mathcal{K}}(\overline{Y})} h(\overline{z_2}).\end{aligned}$$

That is, h is an $\approx$-morphism. Hence, h admits the uniquely determined homomorphic extension $h^\sharp\colon \mathbf{F}_{\mathcal{K}}(\overline{X}) \to \mathbf{F}_{\mathcal{K}}(\overline{Y})$. Moreover,

$$\theta_{h^\sharp}\big(\overline{y_i}, \overline{y_j}\big) = h^\sharp(\overline{y_i}) \approx^{\mathbf{F}_{\mathcal{K}}(\overline{Y})} h^\sharp(\overline{y_j}) = \overline{y} \approx^{\mathbf{F}_{\mathcal{K}}(\overline{Y})} \overline{y} = 1,$$

for $1 \le i, j \le n$. Therefore, $R \subseteq \theta_{h^\sharp}$. As a consequence, $\theta(R) \subseteq \theta_{h^\sharp}$. Thus,

$$\begin{aligned}&\theta(R)\big(p^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y_1}, \ldots, \overline{y_n}),\\ &q^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y_1}, \ldots, \overline{y_n})\big)\\ &\quad\le h^\sharp\big(p^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y_1}, \ldots)\big)\\ &\quad\approx^{\mathbf{F}_{\mathcal{K}}(\overline{Y})} h^\sharp\big(q^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y_1}, \ldots)\big)\\ &\quad= p^{\mathbf{F}_{\mathcal{K}}(\overline{Y})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y}, \ldots, \overline{y})\\ &\quad\approx^{\mathbf{F}_{\mathcal{K}}(\overline{Y})} q^{\mathbf{F}_{\mathcal{K}}(\overline{Y})}(\overline{x_1}, \ldots, \overline{x_m}, \overline{y}, \ldots, \overline{y})\\ &\quad= \theta_{\mathcal{K}}(Y)\big(p(x_1, \ldots, x_m, y, \ldots, y),\\ &q(x_1, \ldots, x_m, y, \ldots, y)\big)\\ &\quad= \|p(x_1, \ldots, x_m, y, \ldots, y)\\ &\quad\approx q(x_1, \ldots, x_m, y, \ldots, y)\|_{\mathcal{K}},\end{aligned}$$

is true for arbitrary terms $p, q \in T(X)$. □

Lemma 3 *Let* $\mathcal{K}$ *be a variety of* **L***-algebras and let* $\mathbf{M} \in \mathcal{K}$. *Then we have*

$$\begin{aligned}&\phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{b})\big) \otimes \theta\big(p^{\mathbf{M}}(\mathfrak{c}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\\ &\quad\le \textstyle\bigwedge_{n\in\mathbb{N}_0}\big(\theta(\mathfrak{a}, \mathfrak{c}) \otimes \phi(\mathfrak{c}, \mathfrak{b})\big)^n \to \big(\phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b})\big)\\ &\qquad\otimes \theta\big(p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\big) \qquad (12)\end{aligned}$$

for every term $p(x, y, z) \in T(X)$, *elements* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in M$, *and* $\theta, \phi \in \mathrm{Con}(\mathbf{M})$.

Proof For brevity, put $d = \bigwedge_{n\in\mathbb{N}_0}\big(\theta(\mathfrak{a}, \mathfrak{c}) \otimes \phi(\mathfrak{c}, \mathfrak{b})\big)^n$. Since every term has only finitely many occurrences of a variable, it follows that

$$d = \textstyle\bigwedge_{n\in\mathbb{N}_0}\big(\theta(\mathfrak{a}, \mathfrak{c})^n \otimes \phi(\mathfrak{c}, \mathfrak{b})^n\big) \le \theta(\mathfrak{a}, \mathfrak{c})^{|p|_x} \otimes \phi(\mathfrak{c}, \mathfrak{b})^{|p|_y}.$$

Moreover, using the adjointness property, monotony of $\otimes$, transitivity of θ, ϕ together with (9), we have

$$\begin{aligned}&\phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{b})\big) \otimes \theta\big(p^{\mathbf{M}}(\mathfrak{c}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\\ &\quad\le d \to \big(\theta(\mathfrak{a}, \mathfrak{c})^{|p|_x} \otimes \phi(\mathfrak{c}, \mathfrak{b})^{|p|_y} \otimes\ \phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{b})\big)\\ &\qquad\otimes \theta\big(p^{\mathbf{M}}(\mathfrak{c}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\big)\\ &\quad\le d \to \big(\theta\big(p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b}), p^{\mathbf{M}}(\mathfrak{c}, \mathfrak{c}, \mathfrak{b})\big) \otimes \theta\big(p^{\mathbf{M}}(\mathfrak{c}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\\ &\qquad\otimes \phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{b})\big) \otimes \phi\big(p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{b}), p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b})\big)\big)\\ &\quad\le d \to \big(\phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b})\big) \otimes \theta\big(p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\big).\end{aligned}$$

Hence, the inequality holds. □

Proof of Theorem from Section 2.

"$\le$": Obviously, $\mathrm{Per}(\mathcal{K}) \le \mathrm{Per}(\mathbf{M})$ for every $\mathbf{M} \in \mathcal{K}$. Take principal congruences $\theta({}^1/\langle\overline{x}, \overline{y}\rangle), \theta({}^1/\langle\overline{y}, \overline{z}\rangle) \in \mathrm{Con}(\mathbf{F}_{\mathcal{K}}(\overline{X}))$. Now using (11), we obtain

$$\begin{aligned}&\mathrm{Per}(\mathcal{K}) \le \mathrm{Per}(\mathbf{F}_{\mathcal{K}}(\overline{X})) \le \mathrm{Per}(\theta({}^1/\langle\overline{x}, \overline{y}\rangle), \theta({}^1/\langle\overline{y}, \overline{z}\rangle))\\ &\quad= S(\theta({}^1/\langle\overline{x}, \overline{y}\rangle) \circ^\omega \theta({}^1/\langle\overline{y}, \overline{z}\rangle), \theta({}^1/\langle\overline{y}, \overline{z}\rangle) \circ \theta({}^1/\langle\overline{x}, \overline{y}\rangle))\\ &\quad\le \big(\theta({}^1/\langle\overline{x}, \overline{y}\rangle) \circ^\omega \theta({}^1/\langle\overline{y}, \overline{z}\rangle)\big)(\overline{x}, \overline{z})\\ &\quad\to \big(\theta({}^1/\langle\overline{y}, \overline{z}\rangle) \circ \theta({}^1/\langle\overline{x}, \overline{y}\rangle)\big)(\overline{x}, \overline{z})\\ &\quad\le \textstyle\bigwedge_{n\in\mathbb{N}_0}\big(\theta({}^1/\langle\overline{x}, \overline{y}\rangle)(\overline{x}, \overline{y}) \otimes \theta({}^1/\langle\overline{y}, \overline{z}\rangle)(\overline{y}, \overline{z})\big)^n\\ &\quad\to \textstyle\bigvee_{p\in T(X)}\big(\theta({}^1/\langle\overline{y}, \overline{z}\rangle)\big(\overline{x}, p^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x}, \overline{y}, \overline{z})\big)\\ &\quad\otimes \theta({}^1/\langle\overline{x}, \overline{y}\rangle)\big(p^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x}, \overline{y}, \overline{z}), \overline{z}\big)\big)\\ &\quad= \textstyle\bigvee_{p\in T(X)}\big(\theta({}^1/\langle\overline{y}, \overline{z}\rangle)\big(\overline{x}, p^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x}, \overline{y}, \overline{z})\big)\\ &\quad\otimes \theta({}^1/\langle\overline{x}, \overline{y}\rangle)\big(p^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x}, \overline{y}, \overline{z}), \overline{z}\big)\big)\\ &\quad\le \textstyle\bigvee_{p\in T(X)}\big(||p(x, y, y) \approx x||_{\mathcal{K}} \otimes ||p(x, x, y) \approx y||_{\mathcal{K}}\big).\end{aligned}$$

"$\ge$": Take arbitrary $\mathbf{M} \in \mathcal{K}$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in M$, and $\theta, \phi \in \mathrm{Con}(\mathbf{M})$. The properties of free **L**-algebras together with (10) and (12) yield

$$\begin{aligned}&\textstyle\bigvee_{p\in T(X)}\big(||p(x, y, y) \approx x||_{\mathcal{K}} \otimes ||p(x, x, y) \approx y||_{\mathcal{K}}\big)\\ &\quad= \textstyle\bigvee_{p\in T(X)}\big(\theta_{\mathcal{K}}(X)(x, p(x, y, y))\\ &\qquad\otimes \theta_{\mathcal{K}}(X)(p(x, x, y), y)\big)\\ &\quad\le \textstyle\bigvee_{p\in T(X)}\big(\phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{b}, \mathfrak{b})\big) \otimes \theta\big(p^{\mathbf{M}}(\mathfrak{c}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\big)\\ &\quad\le \textstyle\bigvee_{p\in T(X)}\Big(\bigwedge_{n\in\mathbb{N}_0}\big(\theta(\mathfrak{a}, \mathfrak{c}) \otimes \phi(\mathfrak{c}, \mathfrak{b})\big)^n\\ &\quad\to \big(\phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b})\big) \otimes \theta\big(p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\big)\Big)\\ &\quad\le \textstyle\bigwedge_{n\in\mathbb{N}_0}\big(\theta(\mathfrak{a}, \mathfrak{c}) \otimes \phi(\mathfrak{c}, \mathfrak{b})\big)^n\\ &\quad\to \textstyle\bigvee_{p\in T(X)}\big(\phi\big(\mathfrak{a}, p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b})\big) \otimes \theta\big(p^{\mathbf{M}}(\mathfrak{a}, \mathfrak{c}, \mathfrak{b}), \mathfrak{b}\big)\big)\\ &\quad\le \textstyle\bigwedge_{n\in\mathbb{N}_0}\big(\theta(\mathfrak{a}, \mathfrak{c}) \otimes \phi(\mathfrak{c}, \mathfrak{b})\big)^n \to (\phi \circ \theta)(\mathfrak{a}, \mathfrak{b}).\end{aligned}$$

Thus, we have

$$\bigvee_{p\in T(X)}\big(||p(x,y,y)\approx x||_{\mathcal{K}}\otimes ||p(x,x,y)\approx y||_{\mathcal{K}}\big)$$
$$\leq \bigwedge_{\mathbf{M}\in\mathcal{K}}\bigwedge_{\theta,\phi\in\mathrm{Con}(\mathbf{M})}\bigwedge_{\mathfrak{a},\mathfrak{b}\in M}\big((\theta\circ^{\omega}\phi)(\mathfrak{a},\mathfrak{b})$$
$$\rightarrow (\phi\circ\theta)(\mathfrak{a},\mathfrak{b})\big)=\mathrm{Per}(\mathcal{K}),$$

proving the claim. □

Remarks Recall that for $\mathbf{L}$ being the two-element Boolean algebra, we obtain exactly the same permutability characterization known from the classical case. Indeed, we have that $\theta\circ\phi=\phi\circ\theta$ iff $\theta\circ\phi=\theta\circ^{\omega}\phi\subseteq\phi\circ\theta$ iff $S(\theta\circ^{\omega}\phi,\phi\circ\theta)=1$. Hence, $\mathcal{K}$ is congruence permutable iff $\mathrm{Per}(\mathcal{K})=1$, i.e. iff there is a ternary term $p(x,y,z)$ such that $||p(x,y,y)\approx x||_{\mathcal{K}}=1$ and $||p(x,x,y)\approx y||_{\mathcal{K}}=1$ on account of (8).

Suppose $\mathbf{L}=\langle L,\wedge,\vee,\otimes,\rightarrow,0,1\rangle$ is a complete residuated lattice, where $\otimes$ coincides with $\wedge$ ($\mathbf{L}$ is a complete Heyting algebra). Since then $\otimes$ is idempotent, the definition of $\mathrm{Per}(\theta,\phi)$ simplifies to $\mathrm{Per}(\theta,\phi)=S(\theta\circ\phi,\phi\circ\theta)$, which is a natural generalization of $\theta\circ\phi\subseteq\phi\circ\theta$ in fuzzy setting. Hence, the degree of equality $E(\theta\circ\phi,\phi\circ\theta)$ is equal to $\mathrm{Per}(\theta,\phi)\wedge\mathrm{Per}(\phi,\theta)$. The degree $\mathrm{Per}(\mathcal{K})$ can then be interpreted as a lower estimation of the degree to which $\theta\circ\phi$ and $\phi\circ\theta$ are similar for all $\theta,\phi\in\mathbf{M}$, $\mathbf{M}\in\mathcal{K}$. Clearly, if $\mathrm{Per}(\mathcal{K})=1$, then $\theta\circ\phi=\phi\circ\theta$ for all congruences $\theta,\phi\in\mathrm{Con}(\mathbf{M})$, where $\mathbf{M}\in\mathcal{K}$. Hence, for $\otimes=\wedge$ the meaning of $\mathrm{Per}(\mathcal{K})$ corresponds well with the classical permutability.

The situation for non-idempotent $\otimes$ is not so straightforward. Note that the $\circ^{\omega}$-composition has been defined to avoid technical problems with the sensitivity of term functions. The interpretation of the truth degree $(R_1\circ^{\omega}R_2)(\mathfrak{a},\mathfrak{b})$ is interesting for $\mathbf{L}$ being a BL-algebra (prelinear and divisible residuated lattice, see [1,12]) on the unit interval $[0,1]$ with $\wedge$ and $\vee$ being the minimum and the maximum, respectively. In such a case, $\otimes$ is a continuous t-norm and for each $a\in L$, $\bigwedge_{n\in\mathbb{N}_0}a^n$ is an idempotent (namely, the greatest idempotent which is less or equal to $a\in L$, see [5]). Thus, the idempotents of L still play an important role, because $(R_1\circ^{\omega}R_2)(\mathfrak{a},\mathfrak{b})$ is a supremum of idempotents. For $\otimes$ being a *continuous Archimedean t-norm* (0, 1 are its only idempotents), (5) simplifies as follows:

$$(R_1\circ^{\omega}R_2)(\mathfrak{a},\mathfrak{b})=\begin{cases}1 & \text{if there is } \mathfrak{c}\in U \text{ such that}\\ & R_1(\mathfrak{a},\mathfrak{c})=R_2(\mathfrak{c},\mathfrak{b})=1,\\ 0 & \text{otherwise.}\end{cases}$$

That is, $R_1\circ^{\omega}R_2$ corresponds with the bivalent relation ${}^1/R_1\circ{}^1/R_2$. As a consequence, $\mathrm{Per}(\mathcal{K})=1$ iff for every $\mathbf{M}\in\mathcal{K}$ and $\theta,\phi\in\mathrm{Con}(\mathbf{M})$ we have ${}^1/\theta\circ{}^1/\phi\subseteq{}^1/(\phi\circ\theta)$.

Example Using the results on fuzzy equational logic [2], one can estimate the permutability degree of a given variety. For instance, let us have a type $F=\{\cdot,{}^{-1},1\}$, where $\cdot$ is a binary function symbol, ${}^{-1}$ is a unary function symbol, and 1 is a constant (nullary function symbol). A couple $\langle t\approx t',a\rangle$, where $t\approx t'$ is an identity of type F and $a\in L$, is called a *graded identity.* The intuitive meaning of $\langle t\approx t',a\rangle$ is "t equals t' in degree (at least) a". An $\mathbf{L}$-algebra $\mathbf{M}$ *satisfies* $\langle t\approx t',a\rangle$ if $||t\approx t'||_{\mathbf{M},v}\geq a$ for every valuation $v: X\rightarrow M$. Now suppose we are given the following graded identities:

G1: $\langle x\cdot(y\cdot z)\approx(x\cdot y)\cdot z,a\rangle$,
G2: $\langle x\cdot 1\approx x,b\rangle$, $\langle 1\cdot x\approx x,b\rangle$,
G3: $\langle x\cdot x^{-1}\approx 1,c\rangle$, $\langle x^{-1}\cdot x\approx 1,c\rangle$.

The class $\mathcal{K}$ of all $\mathbf{L}$-algebras satisfying G1–G3 is a variety, see [3]. It is easy to see that $\mathcal{K}$ contains, among other $\mathbf{L}$-algebras, all groups endowed with compatible $\mathbf{L}$-equalities. In addition to that, the coarseness of $\mathcal{K}$ is determined by the truth degrees $a,b,c\in L$. Namely, for $a=b=c=0$, $\mathcal{K}$ is the class of all $\mathbf{L}$-algebras. On the other hand, if $a=b=c=1$, then $\mathcal{K}$ is the class of all groups with $\mathbf{L}$-equalities, i.e. $\mathcal{K}$ is the class of all groups considered as $\mathbf{L}$-algebras. For the ternary term $x\cdot(y^{-1}\cdot z)$, we can use deduction rules of fuzzy equational logic to estimate the provability degree of identities $x\cdot(y^{-1}\cdot y)\approx x$, $x\cdot(x^{-1}\cdot y)\approx y$ from G1–G3. We have

1. $\langle y^{-1}\cdot y\approx 1,c\rangle$	G3, substitution
2. $\langle x\cdot(y^{-1}\cdot y)\approx x\cdot 1,c\rangle$	by replacement on 1.
3. $\langle x\cdot 1\approx x,b\rangle$	G2
4. $\langle x\cdot(y^{-1}\cdot y)\approx x,c\otimes b\rangle$	by transitivity on 2., 3.

and

1. $\langle x\cdot x^{-1}\approx 1,c\rangle$	G3
2. $\langle (x\cdot x^{-1})\cdot y\approx 1\cdot y,c\rangle$	by replacement on 1.
3. $\langle 1\cdot y\approx y,b\rangle$	G2, substitution
4. $\langle (x\cdot x^{-1})\cdot y\approx y,c\otimes b\rangle$	by transitivity on 2., 3.
5. $\langle x\cdot(x^{-1}\cdot y)$ $\approx(x\cdot x^{-1})\cdot y,a\rangle$	G1, substitution
6. $\langle x\cdot(x^{-1}\cdot y)\approx y,a\otimes c\otimes b\rangle$	by transitivity on 5., 4.

As a consequence of the completeness of fuzzy equational logic [2], it follows that $||x\cdot(y^{-1}\cdot y)\approx x||_{\mathcal{K}}\geq b\otimes c$ and $||x\cdot(x^{-1}\cdot y)\approx y||_{\mathcal{K}}\geq a\otimes b\otimes c$. Hence, $\mathcal{K}$ is congruence permutable in degree at least $a\otimes b^2\otimes c^2$.

Remarks One can proceed in a similar way to obtain further graded properties of classes of $\mathbf{L}$-algebras (modularity, distributivity, regularity of congruences, etc.) and to establish graded Mal'cev-like characterizations. An interesting question is whether it is possible to characterize a large scale of graded properties of varieties at once by generalizing the well-known result of Taylor [18], see also related paper [15].

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