DIRECT LIMITS AND REDUCED PRODUCTS OF ALGEBRAS WITH FUZZY EQUALITIES

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Abstract. We study direct limits and reduced products of algebras with fuzzy equalities. On the one hand, algebras with fuzzy equalities are natural fuzzy structures that disallow to map similar arguments to dissimilar ones. On the other hand, they are exactly the semantic structures of the equational fragment of first-order fuzzy logic. In this paper we propose generalizations of direct limits and reduced products and point out those properties which are not interesting in the classical (bivalent) case, but which seem to be of a crucial importance when considering the quasivarieties of algebras with fuzzy equalities.

1. Introduction

There were various efforts to extend the notion of an algebra. For instance, N. Weaver [20, 21] introduced so-called metric algebras which result by equipping a classical algebra with a metric defined on its universe set. The aim of the metric is to express "closeness" of elements. Using the notion of equicontinuity, metric algebras can represent structures where each function maps pairwise close arguments to close results. Our paper is connected with another extension of an algebra which also formalizes the requirement of having functions mapping close elements to close ones but unlike the metric algebras, our extension is developed in the context of fuzzy logic and does not utilize notions like metric and equicontinuity. In the framework of fuzzy logic, one can formalize "closeness" of elements by so-called fuzzy equivalence relation called also similarity relation. The concept of functions preserving similarity leads to the notion of an algebra with fuzzy equality [5]: an algebra with fuzzy equality is a set with operations on it that is equipped with similarity $\approx$ (a particular fuzzy equivalence relation) such that each operation f is in an appropriate sense compatible with $\approx$. The compatibility ensures that each f yields similar results if applied to pairwise similar arguments.

In addition to the motivation described above, algebras with fuzzy equalities are connected to fuzzy logic in narrow sense (mathematical fuzzy logic) [3, 9, 10, 13, 14, 17]. Namely, algebras with fuzzy equalities represent the semantic (fuzzy) structures of the equational fragment of first-order fuzzy logic. Note that recently fuzzy logic [9, 10, 13, 14] has been profoundly developed. The initial results on algebras with fuzzy equalities [2, 4] showed their nice logical and algebraic properties. Namely, in [2] the author presented a syntactico-semantically complete calculus for reasoning with fuzzy sets of equalities while [4] showed an analogy of the well-known Birkhoff's variety theorem—varieties of algebras with fuzzy equalities are the model classes of fuzzy sets of identities. The present paper is a continuation of [5], where we introduced the basic structural notions, and it tries to shed more light on the constructions which are vital for the development of quasivariety theory in fuzzy setting, see [6, 19].

In fuzzy logic, an important role is played by the chosen structure of truth degrees (or by a whole class of structures of truth degrees). Most of the results on fuzzy logic use residuated lattices as basic structures of truth degrees (even though there are approaches which use noncommutative monoidal structures, but we will not go into this). Residuated lattices, introduced in the 1930s in ring theory, were introduced into the context of fuzzy logic by Goguen [11, 12]. Fundamental contribution to formal fuzzy logic using residuated lattices as the structures of truth values is due to Pavelka [18]. Later on, various logical calculi were investigated using residuated lattices or particular types of residuated lattices. A thorough information about the role of residuated lattices in fuzzy logic can be obtained from monographs [3, 10, 13, 14]. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1,
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid,
(iii) $\langle \otimes, \rightarrow\rangle$ is an adjoint pair, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for each $a, b, c \in L$.

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Particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras (see [14, 15]). In our development we use complete residuated lattices as the basic structures of truth degrees.

Let us now stress the main differences between (classical) algebras and algebras with fuzzy equalities. First, when dealing with algebras with fuzzy equalities we use an explicit structure of truth degrees which is generally much weaker than the two-element Boolean algebra being used implicitly in the (classical) universal algebra. Second, an algebra with fuzzy equality has a nontrivial lattice-valued relational part (similarity relation) which satisfies the compatibility condition. Consequently, structural constructions for algebras with fuzzy equalities have to take care of both the functional part and the relational part. Third, the nontrivial relational part and the general structure of truth values allow us to define the notion of a validity degree of an identity. We can thus consider classes of algebras with fuzzy equalities, where certain identities are satisfied to given degrees, etc. Our extension of ordinary algebras can be also useful to get a deeper insight into the classical structural notions—some classical results generalize for algebras with fuzzy equalities in the full scope (i.e. for any complete residuated lattices taken as the structure of truth degrees), however, some results do not. The latter case is especially interesting because one can identify the explicit requirements on structures of truth degrees which are essential.

Direct limits and reduced products might be interesting also for the fuzzy logic itself. So far, the research in fuzzy logic has been focused almost exclusively on the aspects motivated by the proof theory (structures of truth degrees, semantic consequence, provability, completeness, etc.) Not much attention has been paid to model-theoretical properties of fuzzy structures. In [8], the authors present a generalization of ultraproducts for structures equipped with relations whose truth degrees form a compact Hausdorff space. In fuzzy case, however, there is only a small effort in studying properties and applications of generalized reduced products. As an exception, in [13] the author presents an approach to ultraproducts in fuzzy setting which is based on the ideas of [8]. An analogous result [23] deals with ultraproducts in the context of Pavelka-style fuzzy logic. In this respect, the present paper contributes to model theory for fuzzy logic.

This paper is organized as follows. In Section 2 we present the preliminaries. Section 3 focuses on the direct limits of algebras with fuzzy equalities. In Section 4 we discuss reduced products of algebras with fuzzy equalities and give some remarks on the correlation of the strengthened constructions.

2. Preliminaries

In the sequel we briefly recall basic notions of fuzzy logic, fuzzy sets, and algebras with fuzzy equalities. For a detailed description we refer to [5]. In what follows, **L** always refers to a complete residuated lattice. For **L** we define notions of an **L**-set, **L**-relation, etc. All properties of complete residuated lattices used in the sequel are well known and can be found in any of the above mentioned monographs. Note that the paper contains several examples in which we use the complete residuated $\mathbf{L} = \langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ defined on the real unit interval (unless otherwise stated, the particular definitions of $\otimes$ and $\rightarrow$ will not play any role). For brevity we denote $\mathbf{L} = \langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ simply by $\mathbf{L} = [0,1]$.

An **L**-*set* A (or fuzzy set with truth degrees in **L**) on a universe set U is a mapping $A\colon U \rightarrow L$, $A(\mathfrak{u}) \in L$ being interpreted as the truth degree of "element $\mathfrak{u}$ belongs to A". For every **L**-set $A\colon U \rightarrow L$, the *support set of* A, denoted by $\mathrm{Supp}(A)$, is defined by $\mathrm{Supp}(A) = \{\mathfrak{u} \in U \mid A(\mathfrak{u}) > 0\}$. An **L**-set A is called *finite* if $\mathrm{Supp}(A)$ is finite. For **L**-sets A and B on the same universe set U we write $A \subseteq B$ iff $A(\mathfrak{u}) \leq B(\mathfrak{u})$ for each $\mathfrak{u} \in U$; and $A = B$ iff $A \subseteq B$ and $B \subseteq A$. An **L**-set A in U is called *crisp* if $A(\mathfrak{u}) \in \{0,1\}$ for each $\mathfrak{u} \in U$. If there is no danger of confusion, we sometimes identify the classical sets with crisp **L**-sets. A *binary* **L**-*relation* R on U is an **L**-set on the universe set $U \times U$, i.e. it is a mapping $R\colon U \times U \rightarrow L$. A binary **L**-relation $R'\colon U \times U \rightarrow L$ is called a *restriction of* $R\colon U \times U \rightarrow L$, if $R'(\mathfrak{u},\mathfrak{v}) \neq 0$ implies $R'(\mathfrak{u},\mathfrak{v}) = R(\mathfrak{u},\mathfrak{v})$ for all $\mathfrak{u},\mathfrak{v} \in U$. A restriction R' of R is called *a finite restriction* if R' is a finite **L**-relation.

An **L**-*equivalence* (fuzzy equivalence, similarity) relation E on a set U is a binary **L**-relation on U satisfying $E(\mathfrak{u},\mathfrak{u}) = 1$ (reflexivity), $E(\mathfrak{u},\mathfrak{v}) = E(\mathfrak{v},\mathfrak{u})$ (symmetry) and $E(\mathfrak{u},\mathfrak{v}) \otimes E(\mathfrak{v},\mathfrak{w}) \leq E(\mathfrak{u},\mathfrak{w})$ (transitivity) for every $\mathfrak{u},\mathfrak{v},\mathfrak{w} \in U$. An **L**-equivalence on U where $E(\mathfrak{u},\mathfrak{v}) = 1$ implies $\mathfrak{u} = \mathfrak{v}$ is called an **L**-*equality* (fuzzy equality). Given an **L**-equivalence E, the degree $E(\mathfrak{u},\mathfrak{v}) \in L$ can be interpreted as the truth degree of proposition "$\mathfrak{u}$ and $\mathfrak{v}$ are similar (E-equivalent)". The above mentioned conditions of reflexivity, symmetry, and transitivity are exactly the semantic representations of the well-known equivalence axioms which are required to be satisfied in degree 1 (the greatest element of **L**) [3]. In the sequel, we shall define certain algebraic constructions which involve factor sets determined by **L**-equivalences: for any **L**-equivalence E on U let ${}^1E$ be the binary relation on U such that $\langle \mathfrak{u},\mathfrak{v}\rangle \in {}^1E$ iff $E(\mathfrak{u},\mathfrak{v}) = 1$. It is easily seen that ${}^1E$ is a classical equivalence relation. Thus, one may consider the factor set $U/{}^1E$ of U by ${}^1E$. For brevity we denote $U/{}^1E$ by U/E and call it the *factor set of* U *by* E.

Moreover, we let $[\mathfrak{u}]_E$ denote the factor class of U by ${}^1E$ which contains $\mathfrak{u} \in U$, i.e. $[\mathfrak{u}]_E = \{\mathfrak{v} \in U \mid E(\mathfrak{u},\mathfrak{v}) = 1\}$. For each $[\mathfrak{u}]_E$ and $[\mathfrak{v}]_E$ we put $[\mathfrak{u}]_E \approx^{U/E} [\mathfrak{v}]_E = E(\mathfrak{u},\mathfrak{v})$. Here $\approx^{U/E}$ is a well-defined **L**-equality on U/E. Indeed, applying symmetry and transitivity of E, if $\mathfrak{u}' \in [\mathfrak{u}]_E$, and $\mathfrak{v}' \in [\mathfrak{v}]_E$ then $E(\mathfrak{u},\mathfrak{u}') = E(\mathfrak{v},\mathfrak{v}') = 1$, i.e. $E(\mathfrak{u},\mathfrak{v}) = E(\mathfrak{u},\mathfrak{v}) \otimes E(\mathfrak{u},\mathfrak{u}') \otimes E(\mathfrak{v},\mathfrak{v}') \leq E(\mathfrak{u}',\mathfrak{v}')$. Analogously, $E(\mathfrak{u}',\mathfrak{v}') \leq E(\mathfrak{u},\mathfrak{v})$. Therefore, $\approx^{U/E}$ is a well-defined binary **L**-relation on U/E. Moreover, for any $\mathfrak{u},\mathfrak{v} \in U$ with $[\mathfrak{u}]_E \approx^{U/E} [\mathfrak{v}]_E = 1$ we have $E(\mathfrak{u},\mathfrak{v}) = 1$. That is, $\mathfrak{u} \in [\mathfrak{v}]_E$, i.e. $[\mathfrak{u}]_E = [\mathfrak{v}]_E$. The transitivity, symmetry, and reflexivity of $\approx^{U/E}$ follow by properties of E.

A mapping $f\colon U^n \to U$, $n \in \mathbb{N}$, is *compatible with a binary* **L**-*relation* R *on* U if for any $\mathfrak{u}_1,\mathfrak{v}_1,\ldots,\mathfrak{u}_n,\mathfrak{v}_n \in U$ we have

$$R(\mathfrak{u}_1,\mathfrak{v}_1) \otimes \cdots \otimes R(\mathfrak{u}_n,\mathfrak{v}_n) \leq R(f(\mathfrak{u}_1,\ldots,\mathfrak{u}_n), f(\mathfrak{v}_1,\ldots,\mathfrak{v}_n)). \tag{1}$$

Compatibility, being the semantic representation of the compatibility (congruence) axiom, has a natural verbal description: it says "if $\mathfrak{u}_1$ and $\mathfrak{v}_1$ are R-related and $\cdots$ and $\mathfrak{u}_n$ and $\mathfrak{v}_n$ are R-related then $f(\mathfrak{u}_1,\ldots,\mathfrak{u}_n)$ and $f(\mathfrak{v}_1,\ldots,\mathfrak{v}_n)$ are R-related".

We are going to introduce the notion of an algebra with fuzzy equality. As usual, by a *type* we mean a collection F of function symbols $f \in F$ together with their arities. Let $t, s, \ldots$ and $t \approx t', s \approx s', \ldots$ denote terms (defined as usual) and identities (of a given type F), respectively. The set of all terms of type F in variables X will be denoted by $T(X)$. An *algebra with* **L**-*equality* (shortly an **L**-algebra) of type F is a triplet $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$, where $\langle M, F^{\mathbf{M}}\rangle$ is an (ordinary) algebra of type F (the so-called *skeleton* of **M**) and $\approx^{\mathbf{M}}$ is an **L**-equality on M such that each function $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$. For brevity, the skeleton of **M** will be denoted by ske(**M**). Before presenting further notions, let us stress the role of compatibility. In the classical case, the compatibility axiom is trivially satisfied. In the fuzzy setting, the compatibility might be thought of as a constraint for operations. Ordinary algebras can be interpreted as **L**-algebras. For instance, if **L** is the two-element Boolean algebra, then the notion of an **L**-algebra coincides with that of an (ordinary) algebra with the usual equality—this way our approach generalizes the results of universal algebra. Another way of interpreting an ordinary algebra $\langle M, F^{\mathbf{M}}\rangle$ as an **L**-algebra is as follows: we consider $\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ such that $\approx^{\mathbf{M}}$ is crisp, i.e. $\{\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} \mid \mathfrak{u},\mathfrak{v} \in M\} \subseteq \{0,1\}$ [5].

In classical case, given an algebra **M**, an identity $t \approx t'$ is either valid in **M** or not. In the fuzzy setting, however, an identity can have a general validity degree (i.e. a truth degree from L), not necessary only 0 or 1. 0 and 1 are but two particular validity degrees, namely, the extremal ones. Given an **L**-algebra **M** and a valuation $v\colon X \to M$, the interpretation $\|t\|_{\mathbf{M},v}$ of a term $t \in T(X)$ in **M** under v is defined as usual. For an identity $t \approx t'$ we define the *degree* $\|t \approx t'\|_{\mathbf{M},v} \in L$ *to which* $t \approx t'$ *is true in* **M** *under* v by $\|t \approx t'\|_{\mathbf{M},v} = \|t\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t'\|_{\mathbf{M},v}$. Finally, the *degree* $\|t \approx t'\|_{\mathbf{M}} \in L$ *to which* $t \approx t'$ *is valid in* **M** is defined using the infimum of truth degrees ranging over all valuations: $\|t \approx t'\|_{\mathbf{M}} = \bigwedge_{v:X\to M} \|t \approx t'\|_{\mathbf{M},v}$. It is obvious that for **L** being the two-element Boolean algebra the notion of validity defined in truth degrees coincides with the ordinary one.

Example 2.1. (a) If $T(X) \neq \emptyset$ then we equip $T(X)$ with functions $f^{\mathbf{T}(X)}$ ($f \in F$) such that $f^{\mathbf{T}(X)}(t_1,\ldots,t_n) = f(t_1,\ldots,t_n)$. Let $F^{\mathbf{T}(X)}$ be the collection of all $f^{\mathbf{T}(X)}$'s. In addition to that, we define a binary **L**-relation $\approx^{\mathbf{T}(X)}$ on $T(X)$ by

$$t \approx^{\mathbf{T}(X)} s = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Clearly, $\mathbf{T}(X) = \langle T(X), \approx^{\mathbf{T}(X)}, F^{\mathbf{T}(X)}\rangle$ is an **L**-algebra. $\mathbf{T}(X)$ is called the *term* **L**-*algebra of type* F *in variables* X. Even though the relational part of $\mathbf{T}(X)$ is crisp, term **L**-algebras can be used to get **L**-algebras with nontrivial fuzzy equalities (e.g. by factorization, see below). Term **L**-algebras play an analogous role as the classical term algebras in universal algebra and will be used in further sections.

(b) Let $\mathbf{L} = \langle [0,1], \min, \max, \otimes, \to, 0, 1\rangle$ be the standard Łukasiewicz algebra [3, 13, 14, 17]. That is, the multiplication $\otimes$ and residuum $\to$ are given by $a \otimes b = \max(a+b-1, 0)$, and $a \to b = \min(1-a+b, 1)$. Consider a type which consists of a single binary function symbol $\circ$. We define an **L**-equality $\approx^{\mathbf{M}}$ and an operation $\circ^{\mathbf{M}}$ on a universe set $M = \{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d},\mathfrak{e},\mathfrak{f}\}$ by the following tables:

$\approx^{\mathbf{M}}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{a}$	1	$\frac{3}{4}$	$\frac{3}{4}$	0	$\frac{1}{4}$	0
$\mathfrak{b}$	$\frac{3}{4}$	1	$\frac{3}{4}$	0	$\frac{1}{4}$	0
$\mathfrak{c}$	$\frac{3}{4}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$\mathfrak{d}$	0	0	$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{7}{8}$
$\mathfrak{e}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{4}$
$\mathfrak{f}$	0	0	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{3}{4}$	1

$\circ^{\mathbf{M}}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{d}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{d}$	$\mathfrak{f}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$

One can check that $\circ^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$, i.e. $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, \circ^{\mathbf{M}} \rangle$ is an $\mathbf{L}$-algebra. Note that ske($\mathbf{M}$), being the functional part of $\mathbf{M}$, is idempotent but it is neither associative nor commutative. On the other hand, we have $\|x \circ y \approx y \circ x\|_{\mathbf{M}} = \|x \circ (y \circ z) \approx (x \circ y) \circ z\|_{\mathbf{M}} = \frac{7}{8}$. This can be read: "$\mathbf{L}$-algebra $\mathbf{M}$ is commutative and associative in truth degree $\frac{7}{8}$". For more examples we refer the reader to [5].

In [5] we developed algebraic constructions (morphisms, subalgebras, direct products) respecting both the functional and the (non-trivial) relational part of $\mathbf{L}$-algebras. We briefly summarize the constructions involved in the subsequent development.

An $\mathbf{L}$-algebra $\mathbf{N}$ is a *subalgebra* of an $\mathbf{L}$-algebra $\mathbf{M}$ if (i) ske($\mathbf{N}$) is a subalgebra of ske($\mathbf{M}$), and (ii) $\approx^{\mathbf{N}}$ is a restriction of $\approx^{\mathbf{M}}$ on N. Let $\mathbf{M}, \mathbf{N}$ be $\mathbf{L}$-algebras. A mapping $h\colon M \to N$ satisfying $\mathfrak{a} \approx^{\mathbf{M}} \mathfrak{b} \leq h(\mathfrak{a}) \approx^{\mathbf{N}} h(\mathfrak{b})$ (in the order $\leq$ of $\mathbf{L}$) for all $\mathfrak{a}, \mathfrak{b} \in M$ is called an *$\approx$-morphism.* Thus, an $\approx$-morphism is a mapping which preserves (is compatible with) the $\mathbf{L}$-equalities (the relational parts of $\mathbf{L}$-algebras). An $\approx$-morphism $h\colon M \to N$ is called a *morphism* (*of* $\mathbf{L}$-*algebras*), denoted by $h\colon \mathbf{M} \to \mathbf{N}$, if h is a morphism of ske($\mathbf{M}$) and ske($\mathbf{N}$). Given morphisms $h : \mathbf{M} \to \mathbf{M}'$, $g : \mathbf{M}' \to \mathbf{M}''$, the composed mapping $(h \circ g) : \mathbf{M} \to \mathbf{M}''$ is a morphism. In much the same way as in the classical case, we distinguish special morphisms of $\mathbf{L}$-algebras. A surjective morphism $h\colon \mathbf{M} \to \mathbf{N}$ satisfying $\mathfrak{a} \approx^{\mathbf{M}} \mathfrak{b} = h(\mathfrak{a}) \approx^{\mathbf{N}} h(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in M$ is called an *isomorphism.* $\mathbf{L}$-algebras $\mathbf{M}$ and $\mathbf{N}$ are called *isomorphic,* in symbols $\mathbf{M} \cong \mathbf{N}$, if there is an isomorphism $h\colon \mathbf{M} \to \mathbf{N}$. Observe that for isomorphic $\mathbf{L}$-algebras $\mathbf{M}$ and $\mathbf{N}$ we have $\|t \approx t'\|_{\mathbf{M}} = \|t \approx t'\|_{\mathbf{N}}$ for every identity $t \approx t'$. That is, isomorphic $\mathbf{L}$-algebras cannot be distinguished by the (graded) validity of any identity. Trivially, the identical mapping $\mathrm{id}_M : M \to M$ on the universe set of $\mathbf{M}$ is an isomorphism. Each mapping $h\colon X \to M$ has a uniquely determined *homomorphic extension* $h^\sharp : \mathbf{T}(X) \to \mathbf{M}$ (the term $\mathbf{L}$-algebra $\mathbf{T}(X)$ is defined the same way as in Example 2.1), i.e. $h^\sharp$ is a morphism such that $h(x) = h^\sharp(x)$ for all $x \in X$. An $\mathbf{L}$-relation θ on M is called a *congruence on* $\mathbf{M}$ if (i) θ is an $\mathbf{L}$-equivalence relation on M, (ii) $\approx^{\mathbf{M}} \subseteq \theta$, and (iii) all functions $f^{\mathbf{M}} \in F^{\mathbf{M}}$ are compatible with θ. The congruences on $\mathbf{M}$ form a complete lattice the least and the greatest elements of which are $\approx^{\mathbf{M}}$, and θ such that $\theta(\mathfrak{a}, \mathfrak{b}) = 1$ ($\mathfrak{a}, \mathfrak{b} \in M$), respectively. For brevity, we denote the greatest congruence on $\mathbf{M}$ simply by $M \times M$. For a binary $\mathbf{L}$-relation R on M we denote by $\theta(R)$ the congruence generated by R. Given a congruence θ on $\mathbf{M}$, the $\mathbf{L}$-algebra $\mathbf{M}/\theta = \langle M/\theta, \approx^{\mathbf{M}/\theta}, F^{\mathbf{M}/\theta} \rangle$ is called the *factor* $\mathbf{L}$-*algebra of* $\mathbf{M}$ *modulo* θ, if (i) ske($\mathbf{M}/\theta$) is the factor algebra of ske($\mathbf{M}$) modulo ${}^1\theta$, and (ii) $[\mathfrak{a}]_\theta \approx^{\mathbf{M}/\theta} [\mathfrak{b}]_\theta = \theta(\mathfrak{a}, \mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in M$. For a morphism $h\colon \mathbf{M} \to \mathbf{N}$ we define a congruence θ_h on $\mathbf{M}$ by $\theta_h(\mathfrak{a}, \mathfrak{b}) = h(\mathfrak{a}) \approx^{\mathbf{N}} h(\mathfrak{b})$. For a congruence θ on $\mathbf{M}$, a morphism $h_\theta : \mathbf{M} \to \mathbf{M}/\theta$ defined by $h_\theta(\mathfrak{a}) = [\mathfrak{a}]_\theta$ ($\mathfrak{a} \in M$) is called a *natural morphism.* Given a system $\{\mathbf{M}_i \,|\, i \in I\}$ of $\mathbf{L}$-algebras, a *direct product* is an $\mathbf{L}$-algebra $\prod_{i \in I} \mathbf{M}_i = \langle \prod_{i \in I} M_i, \approx^{\prod_{i \in I} \mathbf{M}_i}, F^{\prod_{i \in I} \mathbf{M}_i} \rangle$, where (i) ske($\prod_{i \in I} \mathbf{M}_i$) is the direct product $\prod_{i \in I} \mathrm{ske}(\mathbf{M}_i)$, and (ii) $\mathfrak{a} \approx^{\prod_{i \in I} \mathbf{M}_i} \mathfrak{b} = \bigwedge_{i \in I} \mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i)$ for all $\mathfrak{a}, \mathfrak{b} \in \prod_{i \in I} M_i$.

The properties of morphisms and congruences of $\mathbf{L}$-algebras are analogous to those of their classical counterparts. For instance, the well-known isomorphism theorems hold for $\mathbf{L}$-algebras in the full scope [3, 5]. On the other hand, some properties of algebras generalize only for particular subclasses of complete residuated lattices. This is the case of e.g. subdirect representation [5]. Nevertheless, in the sequel we shall use another representation of $\mathbf{L}$-algebras. Let $\mathbf{T}(X)$ be a term $\mathbf{L}$-algebra (see Example 2.1), R be a binary $\mathbf{L}$-relation on $T(X)$. Each $\mathbf{L}$-algebra $\mathbf{M}$ such that $\mathbf{M} \cong \mathbf{T}(X)/\theta(R)$ is said to be *presented by* $\langle X, R \rangle$. Moreover, $\mathbf{M}$ is called *finitely presented* if X and R can be chosen so that X is a finite set and R is a finite $\mathbf{L}$-relation. Note that each $\mathbf{L}$-algebra $\mathbf{M}$ is presented by $\langle X, R \rangle$, where X is a suitably large set of variables and R is a binary $\mathbf{L}$-relation on $T(X)$. Indeed, one can consider $|X| \geq |M|$, and a surjective mapping $h\colon X \to M$. Then for the homomorphic extension $h^\sharp : \mathbf{T}(X) \to \mathbf{M}$ of h we have $\mathbf{M} \cong \mathbf{T}(X)/\theta_{h^\sharp}$ due to the first isomorphism theorem [3, 5]. That is, $\mathbf{M}$ is presented by $\langle X, \theta_{h^\sharp} \rangle$.

Remark 2.2. Let us comment some more on the role of complete residuated lattice as the structures of truth degrees. In the definition of algebras with fuzzy equalities, we used truth degrees from a lattice ordered structure (of truth degrees) to express similarity of elements. The condition of compatibility with functions (1) was defined using the multiplication $\otimes$ which can be seen as a particular interpretation of logical connective "conjunction". This suggests that our structures of truth degrees should be particular lattice-ordered monoids. Moreover, in order to consider direct products of algebras with fuzzy equalities, we assumed that the lattice order is complete (otherwise $\mathfrak{a} \approx^{\prod_{i \in I} \mathbf{M}_i} \mathfrak{b}$ would not be defined in general, see the definition of $\prod_{i \in I} \mathbf{M}_i$). Thus, our structures of truth degrees should be complete lattice-ordered monoinds. Each complete residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ fulfills this requirement. On the other hand, one may ask if we actually need the binary operation of residuum $\to$ (interpretation of logical connective "implication"). Even if $\to$ has not been (explicitly) used so far, it is important. For instance, in order to have desirable properties of several algebraic constructions, we need the following relationship between $\otimes$ and $\bigvee$:

$$a \otimes \bigvee_{i\in I} b_i = \bigvee_{i\in I} (a \otimes b_i). \tag{3}$$

Now, the following assertion is true (see [3]): if $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, and $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative, and $a \otimes 1 = 1 \otimes a = a$, where 1 the greatest element of L with respect to the lattice order $\leq$), then there exists a residuum $\rightarrow$ such that $\langle \otimes, \rightarrow\rangle$ is an adjoint pair iff (3) is true for any $a \in L$ and $\{b_i \in L \,|\, i \in I\}$ (namely, one can put $a \rightarrow b = \bigvee\{c \in L \,|\, a \otimes c \leq b\}$). Since the required property (3) is equivalent to the existence of residuum, we develop algebras with fuzzy equalities over complete residuated lattices. In further sections we will take advantage of several properties of residuated lattices which can be found in [3, 10, 13, 14].

3. Direct limits

We begin our development with the definition of a *weak direct family* which generalizes the notion of a direct family known from the (classical) universal algebra. In addition to that, we introduce its strengthened form by postulating an additional condition (which holds true automatically in the ordinary case). Having generalized the basic notions, we introduce direct limits and analyze their properties.

Definition 3.1. A *weak direct family* (of $\mathbf{L}$-algebras) consists of:

(i) a directed index set $\langle I, \leq\rangle$, i.e. $I \neq \emptyset$, and for every $i, j \in I$ there is $k \in I$ such that $i, j \leq k$;
(ii) a family $\{\mathbf{M}_i \,|\, i \in I\}$ of pairwise disjoint $\mathbf{L}$-algebras;
(iii) a family $\{h_{ij} \colon \mathbf{M}_i \to \mathbf{M}_j \,|\, i \leq j\}$ of morphisms, where

$$h_{ii} = \mathrm{id}_{M_i} \qquad \text{for every } i \in I, \tag{4}$$

$$h_{ik} = h_{ij} \circ h_{jk} \qquad \text{for all } i, j, k \in I, \text{where } i \leq j \leq k. \tag{5}$$

A *weak direct family* is called a *direct family* if for every $\mathfrak{a} \in M_i$, and $\mathfrak{b} \in M_j$ there exists $k \in I$ such that $i, j \leq k$, and for each $l \in I$ with $k \leq l$ we have

$$h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) = h_{il}(\mathfrak{a}) \approx^{\mathbf{M}_l} h_{jl}(\mathfrak{b}). \tag{6}$$

Remark 3.2. (a) A (weak) direct family of $\mathbf{L}$-algebras will be denoted simply by $\{\mathbf{M}_i \,|\, i \in I\}$. If there is no danger of confusion, we will not mention the morphisms $h_{ij} \colon \mathbf{M}_i \to \mathbf{M}_j$ explicitly.

(b) In general, there are weak direct families which do not satisfy (6). Take $\mathbf{L} = [0, 1]$ as the structure of truth degrees and a family $\{\mathbf{M}_i \,|\, i \in [0, 1)\}$ of $\mathbf{L}$-algebras, where $\mathbf{M}_i = \langle\{\mathfrak{a}_i, \mathfrak{b}_i\}, \approx^{\mathbf{M}_i}, \emptyset\rangle$, and $\mathfrak{a}_i \approx^{\mathbf{M}_i} \mathfrak{b}_i = i$. Moreover, morphisms $h_{ij} \colon \mathbf{M}_i \to \mathbf{M}_j$ $(i \leq j)$ defined by $h_{ij}(\mathfrak{a}_i) = \mathfrak{a}_j$, $h_{ij}(\mathfrak{b}_i) = \mathfrak{b}_j$ evidently satisfy (4) and (5). Therefore, $\langle [0, 1), \leq\rangle$ together with $\{\mathbf{M}_i \,|\, i \in [0, 1)\}$ and h_{ij}'s is a weak direct family. On the other hand, for $\mathfrak{a}_i, \mathfrak{b}_j \in \bigcup_{m\in[0,1)} M_m$, and every $k \geq i, j$ there is $l > k$ such that

$$h_{ik}(\mathfrak{a}_i) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}_j) = \mathfrak{a}_k \approx^{\mathbf{M}_k} \mathfrak{b}_k = k < l = \mathfrak{a}_l \approx^{\mathbf{M}_l} \mathfrak{b}_l = h_{il}(\mathfrak{a}_i) \approx^{\mathbf{M}_l} h_{jl}(\mathfrak{b}_j),$$

showing that $\{\mathbf{M}_i \,|\, i \in [0, 1)\}$ is not a direct family.

Lemma 3.3. *Let $\{\mathbf{M}_i \,|\, i \in I\}$ be a weak direct family. For every $\mathfrak{a} \in M_i$, $\mathfrak{b} \in M_j$, and arbitrary indices $k, l \in I$ such that $i, j \leq k \leq l$ we have*

$$h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) \leq h_{il}(\mathfrak{a}) \approx^{\mathbf{M}_l} h_{jl}(\mathfrak{b}), \tag{7}$$

$$\bigvee_{k\geq i,j} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) = \bigvee_{m\geq l} h_{im}(\mathfrak{a}) \approx^{\mathbf{M}_m} h_{jm}(\mathfrak{b}). \tag{8}$$

Proof. (7): Using (5) we have $h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) \leq h_{kl}(h_{ik}(\mathfrak{a})) \approx^{\mathbf{M}_l} h_{kl}(h_{jk}(\mathfrak{b})) = h_{il}(\mathfrak{a}) \approx^{\mathbf{M}_l} h_{jl}(\mathfrak{b})$.

(8): Take an index $k_0 \geq i, j$. For $m_0 \in I$ such that $m_0 \geq k_0, l$, we can use (7) to get

$$h_{ik_0}(\mathfrak{a}) \approx^{\mathbf{M}_{k_0}} h_{jk_0}(\mathfrak{b}) \leq h_{im_0}(\mathfrak{a}) \approx^{\mathbf{M}_{m_0}} h_{jm_0}(\mathfrak{b}) \leq \bigvee_{m\geq l} h_{im}(\mathfrak{a}) \approx^{\mathbf{M}_m} h_{jm}(\mathfrak{b}),$$

by which follows the "$\leq$"-part of (8). The converse inequality holds trivially. ☐

Remark 3.4. Let $\mathbf{L}$ be a complete residuated lattice, where each $a \in L$ is compact (i.e., $\mathbf{L}$ is a noetherian lattice, see [7]). Then every weak direct family is a direct family. Indeed, the compactness yields that for any $\mathfrak{a} \in M_i$, and $\mathfrak{b} \in M_j$, there are indices $k_1, \ldots, k_n \geq i, j$ such that $\bigvee_{k\geq i,j} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) = \bigvee_{m=1}^{n} h_{ik_m}(\mathfrak{a}) \approx^{\mathbf{M}_{k_m}} h_{jk_m}(\mathfrak{b})$. Thus, take an index $k \in I$ with $k \geq k_1, \ldots, k_n$. Now (7) gives $h_{ik_m}(\mathfrak{a}) \approx^{\mathbf{M}_{k_m}} h_{jk_m}(\mathfrak{b}) \leq h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b})$ for each $m = 1, \ldots, n$. Therefore, $h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b})$ is the greatest one of all $h_{ik'}(\mathfrak{a}) \approx^{\mathbf{M}_{k'}} h_{jk'}(\mathfrak{b})$ for $k' \geq i, j$. Since for each $l \geq k$ we have $h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) \leq h_{il}(\mathfrak{a}) \approx^{\mathbf{M}_l} h_{jl}(\mathfrak{b})$ by (7), it follows that in fact $h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) = h_{il}(\mathfrak{a}) \approx^{\mathbf{M}_l} h_{jl}(\mathfrak{b})$ for all $l \geq k$ proving (6).

Definition 3.5. For a weak direct family $\{\mathbf{M}_i \,|\, i \in I\}$ let θ_∞ denote the binary $\mathbf{L}$-relation on $\bigcup_{i\in I} M_i$ defined by

$$\theta_\infty(\mathfrak{a},\mathfrak{b}) = \textstyle\bigvee_{k\geq i,j} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) \tag{9}$$

for all $\mathfrak{a} \in M_i, \mathfrak{b} \in M_j$.

Remark 3.6. If $\{\mathbf{M}_i \,|\, i \in I\}$ is a direct family, then (9) can be expressed equivalently without using the general suprema. Indeed, taking into account (6) and (8), for all $\mathfrak{a} \in M_i, \mathfrak{b} \in M_j$ there is $k_0 \geq i,j$ such that

$$\theta_\infty(\mathfrak{a},\mathfrak{b}) = \textstyle\bigvee_{k\geq i,j} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) = \bigvee_{m\geq k_0} h_{im}(\mathfrak{a}) \approx^{\mathbf{M}_m} h_{jm}(\mathfrak{b}) = h_{ik_0}(\mathfrak{a}) \approx^{\mathbf{M}_{k_0}} h_{jk_0}(\mathfrak{b}).$$

Lemma 3.7. *Let $\{\mathbf{M}_i \,|\, i \in I\}$ be a weak direct family. The following are properties of θ_∞:*

(i) $\theta_\infty(\mathfrak{a},\mathfrak{b}) = \theta_\infty\big(h_{il}(\mathfrak{a}), h_{jl}(\mathfrak{b})\big)$ *for all $\mathfrak{a} \in M_i$, $\mathfrak{b} \in M_j$, and $l \geq i,j$;*

(ii) θ_∞ *is an $\mathbf{L}$-equivalence on $\bigcup_{i\in I} M_i$;*

(iii) $\theta_\infty\big(\mathfrak{a}, h_{ik}(\mathfrak{a})\big) = 1$ *for every $\mathfrak{a} \in M_i$, and $k \geq i$;*

(iv) *for every n-ary $f \in F$, $\mathfrak{a}_1 \in M_{i_1}, \mathfrak{b}_1 \in M_{j_1}, \ldots, \mathfrak{a}_n \in M_{i_n}, \mathfrak{b}_n \in M_{j_n}$, and $k \geq i_1, j_1, \ldots, i_n, j_n$ we have*

$$\theta_\infty(\mathfrak{a}_1,\mathfrak{b}_1) \otimes \cdots \otimes \theta_\infty(\mathfrak{a}_n,\mathfrak{b}_n) \leq \theta_\infty\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1),\ldots,h_{i_nk}(\mathfrak{a}_n)), f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1),\ldots,h_{j_nk}(\mathfrak{b}_n))\big).$$

Proof. (i): Clearly, (5) and (8) give

$$\begin{aligned}\theta_\infty(\mathfrak{a},\mathfrak{b}) &= \textstyle\bigvee_{k\geq i,j} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) = \bigvee_{m\geq l} h_{im}(\mathfrak{a}) \approx^{\mathbf{M}_m} h_{jm}(\mathfrak{b}) = \\ &= \textstyle\bigvee_{m\geq l} h_{lm}(h_{il}(\mathfrak{a})) \approx^{\mathbf{M}_m} h_{lm}(h_{jl}(\mathfrak{b})) = \theta_\infty(h_{il}(\mathfrak{a}), h_{jl}(\mathfrak{b})).\end{aligned}$$

(ii): We have $\theta_\infty(\mathfrak{a},\mathfrak{a}) = \bigvee_{k\geq i} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{ik}(\mathfrak{a}) = 1$ for every $\mathfrak{a} \in M_i$, i.e. θ_∞ is reflexive. Symmetry is obvious. It remains to check transitivity. Let $\mathfrak{a} \in M_i$, $\mathfrak{b} \in M_j$, $\mathfrak{c} \in M_k$, and let $l \geq i,j,k$. Furthermore, (7) and (i) together with the monotony of $\otimes$ yield

$$\begin{aligned}\theta_\infty(\mathfrak{a},\mathfrak{b}) \otimes \theta_\infty(\mathfrak{b},\mathfrak{c}) &= \theta_\infty(h_{il}(\mathfrak{a}), h_{jl}(\mathfrak{b})) \otimes \theta_\infty(h_{jl}(\mathfrak{b}), h_{kl}(\mathfrak{c})) = \\ &= \textstyle\bigvee_{m\geq l} h_{im}(\mathfrak{a}) \approx^{\mathbf{M}_m} h_{jm}(\mathfrak{b}) \otimes \bigvee_{n\geq l} h_{jn}(\mathfrak{b}) \approx^{\mathbf{M}_n} h_{kn}(\mathfrak{c}) = \\ &= \textstyle\bigvee_{m,n\geq l} \big(h_{im}(\mathfrak{a}) \approx^{\mathbf{M}_m} h_{jm}(\mathfrak{b}) \otimes h_{jn}(\mathfrak{b}) \approx^{\mathbf{M}_n} h_{kn}(\mathfrak{c})\big) = \\ &= \textstyle\bigvee_{n\geq l} \big(h_{in}(\mathfrak{a}) \approx^{\mathbf{M}_n} h_{jn}(\mathfrak{b}) \otimes h_{jn}(\mathfrak{b}) \approx^{\mathbf{M}_n} h_{kn}(\mathfrak{c})\big) \leq \bigvee_{n\geq l} h_{in}(\mathfrak{a}) \approx^{\mathbf{M}_n} h_{kn}(\mathfrak{c}) = \theta_\infty(\mathfrak{a},\mathfrak{c}).\end{aligned}$$

Hence, θ_∞ is an $\mathbf{L}$-equivalence.

(iii): Let us have $\mathfrak{a} \in M_i$, $k \geq i$. Take $l \in I$ such that $l \geq k, i$. The reflexivity of θ_∞ together with (i) give

$$\theta_\infty(\mathfrak{a}, h_{ik}(\mathfrak{a})) = \theta_\infty\big(h_{il}(\mathfrak{a}), h_{kl}(h_{ik}(\mathfrak{a}))\big) = \theta_\infty(h_{il}(\mathfrak{a}), h_{il}(\mathfrak{a})) = 1.$$

(iv): For an n-ary $f \in F$, arbitrary $\mathfrak{a}_m \in M_{i_m}, \mathfrak{b}_m \in M_{j_m}$ $(m = 1,\ldots,n)$, and $k \geq i_1, j_1, \ldots, i_n, j_n$, we can use the compatibility of $f^{\mathbf{M}_l}$ with $\approx^{\mathbf{M}_l}$ to get

$$\begin{aligned}&\theta_\infty(\mathfrak{a}_1,\mathfrak{b}_1) \otimes \cdots \otimes \theta_\infty(\mathfrak{a}_n,\mathfrak{b}_n) = \textstyle\bigotimes_{m=1}^{n} \bigvee_{k_m\geq k} h_{i_mk_m}(\mathfrak{a}_m) \approx^{\mathbf{M}_{k_m}} h_{j_mk_m}(\mathfrak{b}_m) = \\ &\quad= \textstyle\bigvee_{k_1,\ldots,k_m\geq k} \bigotimes_{m=1}^{n} h_{i_mk_m}(\mathfrak{a}_m) \approx^{\mathbf{M}_{k_m}} h_{j_mk_m}(\mathfrak{b}_m) = \bigvee_{l\geq k} \bigotimes_{m=1}^{n} h_{i_ml}(\mathfrak{a}_m) \approx^{\mathbf{M}_l} h_{j_ml}(\mathfrak{b}_m) \leq \\ &\quad\leq \textstyle\bigvee_{l\geq k} f^{\mathbf{M}_l}\big(h_{i_1l}(\mathfrak{a}_1),\ldots,h_{i_nl}(\mathfrak{a}_n)\big) \approx^{\mathbf{M}_l} f^{\mathbf{M}_l}\big(h_{j_1l}(\mathfrak{b}_1),\ldots,h_{j_nl}(\mathfrak{b}_n)\big) = \\ &\quad= \textstyle\bigvee_{l\geq k} f^{\mathbf{M}_l}\big(h_{kl}(h_{i_1k}(\mathfrak{a}_1)),\ldots,h_{kl}(h_{i_nk}(\mathfrak{a}_n))\big) \approx^{\mathbf{M}_l} f^{\mathbf{M}_l}\big(h_{kl}(h_{j_1k}(\mathfrak{b}_1)),\ldots,h_{kl}(h_{j_nk}(\mathfrak{b}_n))\big) = \\ &\quad= \textstyle\bigvee_{l\geq k} h_{kl}\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1),\ldots,h_{i_nk}(\mathfrak{a}_n))\big) \approx^{\mathbf{M}_l} h_{kl}\big(f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1),\ldots,h_{j_nk}(\mathfrak{b}_n))\big) = \\ &\quad= \theta_\infty\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1),\ldots,h_{i_nk}(\mathfrak{a}_n)), f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1),\ldots,h_{j_nk}(\mathfrak{b}_n))\big),\end{aligned}$$

which is the desired inequality. □

Condition (iv) of Lemma 3.7 is similar to that of compatibility, but in this case, (iv) expresses a compatibility with respect to homomorphic images. Now we define suitable operations on the factorization of $\bigcup_{i\in I} M_i$ by θ_∞.

Definition 3.8. Let $\{\mathbf{M}_i \,|\, i \in I\}$ be a (weak) direct family. $\lim \mathbf{M}_i = \big\langle \big(\bigcup_{i\in I} M_i\big)/\theta_\infty, \approx^{\lim \mathbf{M}_i}, F^{\lim \mathbf{M}_i}\big\rangle$, where

(i) $\big(\bigcup_{i\in I} M_i\big)/\theta_\infty$ is a factorization of $\bigcup_{i\in I} M_i$ by θ_∞,

(ii) $[\mathfrak{a}]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} [\mathfrak{b}]_{\theta_\infty} = \theta_\infty(\mathfrak{a},\mathfrak{b})$ for all $[\mathfrak{a}]_{\theta_\infty}, [\mathfrak{b}]_{\theta_\infty} \in \big(\bigcup_{i\in I} M_i\big)/\theta_\infty$

(iii) $f^{\lim \mathbf{M}_i}\big([\mathfrak{a}_1]_{\theta_\infty},\ldots,[\mathfrak{a}_n]_{\theta_\infty}\big) = \big[f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1),\ldots,h_{i_nk}(\mathfrak{a}_n))\big]_{\theta_\infty}$ for every n-ary $f \in F$, arbitrary $[\mathfrak{a}_1]_{\theta_\infty},\ldots,[\mathfrak{a}_n]_{\theta_\infty} \in \big(\bigcup_{i\in I} M_i\big)/\theta_\infty$ with $\mathfrak{a}_1 \in M_{i_1},\ldots,\mathfrak{a}_n \in M_{i_n}$, and $k \in I$ such that $k \geq i_1,\ldots,i_n$,

is called a *direct limit of a (weak) direct family* $\{\mathbf{M}_i \,|\, i \in I\}$.

Remark 3.9. A direct limit $\lim \mathbf{M}_i$ of a weak direct family $\{\mathbf{M}_i \mid i \in I\}$ is an $\mathbf{L}$-algebra. Obviously, $\approx^{\lim \mathbf{M}_i}$ is an $\mathbf{L}$-equality. It remains to show that each $f^{\lim \mathbf{M}_i}$ is well defined and compatible with $\approx^{\lim \mathbf{M}_i}$. First, we show that $\big[f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n))\big]_{\theta_\infty}$ given by (iii) does not depend on the chosen $k \in I$. Thus, take $k' \in I$ with $k' \geq i_1, \dots, i_n$, and arbitrary $l \geq k, k'$. Lemma 3.7 gives

$$\theta_\infty\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n)), f^{\mathbf{M}_l}(h_{i_1l}(\mathfrak{a}_1), \dots, h_{i_nl}(\mathfrak{a}_n))\big) = \\ = \theta_\infty\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n)), h_{kl}\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n))\big)\big) = 1.$$

That is, $\big[f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n))\big]_{\theta_\infty} = \big[f^{\mathbf{M}_l}(h_{i_1l}(\mathfrak{a}_1), \dots, h_{i_nl}(\mathfrak{a}_n))\big]_{\theta_\infty}$ and analogously for k'. Hence,

$$\big[f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n))\big]_{\theta_\infty} = \big[f^{\mathbf{M}_{k'}}(h_{i_1k'}(\mathfrak{a}_1), \dots, h_{i_nk'}(\mathfrak{a}_n))\big]_{\theta_\infty}.$$

Moreover, $f^{\lim \mathbf{M}_i}\big([\mathfrak{a}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty}\big)$ does not depend on $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ chosen from classes $[\mathfrak{a}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty}$, because for $\mathfrak{b}_m \in [\mathfrak{a}_m]_{\theta_\infty}$, $\mathfrak{b}_m \in M_{j_m}$ $(m = 1, \dots, n)$, and $k \geq i_1, j_1, \dots, i_n, j_n$ we have

$$1 = \theta_\infty(\mathfrak{a}_1, \mathfrak{b}_1) \otimes \cdots \otimes \theta_\infty(\mathfrak{a}_n, \mathfrak{b}_n) \leq \theta_\infty\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n)), f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1), \dots, h_{j_nk}(\mathfrak{b}_n))\big) = \\ = \big[f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n))\big]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} \big[f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1), \dots, h_{j_nk}(\mathfrak{b}_n))\big]_{\theta_\infty}.$$

Therefore, $\big[f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n))\big]_{\theta_\infty} = \big[f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1), \dots, h_{j_nk}(\mathfrak{b}_n))\big]_{\theta_\infty}$, i.e. $f^{\lim \mathbf{M}_i}$ is well defined.

It remains to check the compatibility. Take $f^{\lim \mathbf{M}_i} \in F^{\lim \mathbf{M}_i}$ and $[\mathfrak{a}_1]_{\theta_\infty}, [\mathfrak{b}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty}, [\mathfrak{b}_n]_{\theta_\infty} \in \big(\bigcup_{i \in I} M_i\big)/\theta_\infty$, where $\mathfrak{a}_m \in M_{i_m}$, $\mathfrak{b}_m \in M_{j_m}$ $(m = 1, \dots, n)$. For $k \in I$ such that $k \geq i_1, j_1, \dots, i_n, j_n$, Lemma 3.7 together with the definition of $\approx^{\lim \mathbf{M}_i}$ yield

$$[\mathfrak{a}_1]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} [\mathfrak{b}_1]_{\theta_\infty} \otimes \cdots \otimes [\mathfrak{a}_n]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} [\mathfrak{b}_n]_{\theta_\infty} = \theta_\infty(\mathfrak{a}_1, \mathfrak{b}_1) \otimes \cdots \otimes \theta_\infty(\mathfrak{a}_n, \mathfrak{b}_n) \leq \\ \leq \theta_\infty\big(f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n)), f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1), \dots, h_{j_nk}(\mathfrak{b}_n))\big) = \\ = \big[f^{\mathbf{M}_k}(h_{i_1k}(\mathfrak{a}_1), \dots, h_{i_nk}(\mathfrak{a}_n))\big]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} \big[f^{\mathbf{M}_k}(h_{j_1k}(\mathfrak{b}_1), \dots, h_{j_nk}(\mathfrak{b}_n))\big]_{\theta_\infty} = \\ = f^{\lim \mathbf{M}_i}\big([\mathfrak{a}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty}\big) \approx^{\lim \mathbf{M}_i} f^{\lim \mathbf{M}_i}\big([\mathfrak{b}_1]_{\theta_\infty}, \dots, [\mathfrak{b}_n]_{\theta_\infty}\big).$$

Hence, $\lim \mathbf{M}_i$ is a well-defined $\mathbf{L}$-algebra.

Lemma 3.10. *Let $\lim \mathbf{M}_i$ be the weak direct limit of a family $\{\mathbf{M}_i \mid i \in I\}$. Then for every n-ary $f^{\lim \mathbf{M}_i}$, and $[\mathfrak{a}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty} \in \big(\bigcup_{i \in I} M_i\big)/\theta_\infty$ we have*

$$f^{\lim \mathbf{M}_i}\big([\mathfrak{a}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty}\big) = \big[f^{\mathbf{M}_k}(\mathfrak{a}'_1, \dots, \mathfrak{a}'_n)\big]_{\theta_\infty},$$

where $\mathfrak{a}'_1, \dots, \mathfrak{a}'_n \in M_k$, and $\mathfrak{a}'_m \in [\mathfrak{a}_m]_{\theta_\infty}$ $(m = 1, \dots, n)$.

Proof. Since $\theta_\infty(\mathfrak{a}_m, \mathfrak{a}'_m) = 1$ for each $m = 1, \dots, n$, we can take an index $l \in I$ such that $l \geq k, i_1, \dots, i_n$ and apply Definition 3.8 and Lemma 3.7:

$$\big[f^{\mathbf{M}_k}(\mathfrak{a}'_1, \dots, \mathfrak{a}'_n)\big]_{\theta_\infty} = \big[h_{kl}\big(f^{\mathbf{M}_k}(\mathfrak{a}'_1, \dots, \mathfrak{a}'_n)\big)\big]_{\theta_\infty} = \big[f^{\mathbf{M}_l}(h_{kl}(\mathfrak{a}'_1), \dots, h_{kl}(\mathfrak{a}'_n))\big]_{\theta_\infty} = \\ = \big[f^{\mathbf{M}_l}(h_{i_1l}(\mathfrak{a}_1), \dots, h_{i_nl}(\mathfrak{a}_n))\big]_{\theta_\infty} = f^{\lim \mathbf{M}_i}\big([\mathfrak{a}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty}\big),$$

proving the assertion. □

Definition 3.11. Let $\{\mathbf{M}_i \mid i \in I\}$ be a weak direct family. A family $\{h_i \colon \mathbf{M}_i \to \lim \mathbf{M}_i \mid i \in I\}$ of morphisms, where $h_i(\mathfrak{a}) = [\mathfrak{a}]_{\theta_\infty}$ $(i \in I,\ \mathfrak{a} \in M_i)$ is called the *limit cone of the weak direct family* $\{\mathbf{M}_i \mid i \in I\}$.

Let $\mathbf{N}$ be an $\mathbf{L}$-algebra. A family $\{g_i \colon \mathbf{M}_i \to \mathbf{N} \mid i \in I\}$ of morphisms is said to satisfy the *direct limit property (DLP) with respect to* $\{\mathbf{M}_i \mid i \in I\}$ if $g_i = h_{ij} \circ g_j$ for all $i \leq j$ and for every family $\{g'_i \colon \mathbf{M}_i \to \mathbf{N}' \mid i \in I\}$ of morphisms with $g'_i = h_{ij} \circ g'_j$ for all $i \leq j$, there exists a unique morphism $g \colon \mathbf{N} \to \mathbf{N}'$ such that $g'_i = g_i \circ g$ for every $i \in I$.

Remark 3.12. (a) Every $h_i \colon \mathbf{M}_i \to \lim \mathbf{M}_i$ of the limit cone of $\{\mathbf{M}_i \mid i \in I\}$ is indeed a morphism. Clearly, for all $\mathfrak{a}, \mathfrak{b} \in M_i$ $(i \in I)$ we have

$$\mathfrak{a} \approx^{\mathbf{M}_i} \mathfrak{b} \leq \textstyle\bigvee_{k \geq i} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{ik}(\mathfrak{b}) = \theta_\infty(\mathfrak{a}, \mathfrak{b}) = [\mathfrak{a}]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} [\mathfrak{b}]_{\theta_\infty} = h_i(\mathfrak{a}) \approx^{\lim \mathbf{M}_i} h_i(\mathfrak{b}).$$

Furthermore, for an n-ary $f \in F$, and $\mathfrak{a}_1, \dots, \mathfrak{a}_n \in M_i$:

$$h_i\big(f^{\mathbf{M}_i}(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\big) = \big[f^{\mathbf{M}_i}(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\big]_{\theta_\infty} = \big[f^{\mathbf{M}_i}(h_{ii}(\mathfrak{a}_1), \dots, h_{ii}(\mathfrak{a}_n))\big]_{\theta_\infty} = \\ = f^{\lim \mathbf{M}_i}\big([\mathfrak{a}_1]_{\theta_\infty}, \dots, [\mathfrak{a}_n]_{\theta_\infty}\big) = f^{\lim \mathbf{M}_i}\big(h_i(\mathfrak{a}_1), \dots, h_i(\mathfrak{a}_n)\big),$$

i.e. h_i is a morphism. (iii) of Lemma 3.7 gives $h_i(\mathfrak{a}) = [\mathfrak{a}]_{\theta_\infty} = [h_{ij}(\mathfrak{a})]_{\theta_\infty} = h_j(h_{ij}(\mathfrak{a}))$, i.e. $h_i = h_{ij} \circ h_j$.

(b) If $\langle I, \leq\rangle$ is a finite directed index set, then I has the greatest element. Therefore, every weak direct family $\{\mathbf{M}_i \,|\, i \in I\}$ is a direct family since for every $\mathfrak{a} \in M_i$, and $\mathfrak{b} \in M_j$, (6) is satisfied trivially for k being the greatest element of I. Consequently, $\theta_\infty(\mathfrak{a}, \mathfrak{b}) = h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b})$. Moreover, for $h_k \colon \mathbf{M}_k \to \lim \mathbf{M}_i$ we have

$$\mathfrak{a} \approx^{\mathbf{M}_k} \mathfrak{b} = h_{kk}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{kk}(\mathfrak{b}) = \theta_\infty(\mathfrak{a}, \mathfrak{b}) = h_k(\mathfrak{a}) \approx^{\lim \mathbf{M}_i} h_k(\mathfrak{b}),$$

and for every $[\mathfrak{c}]_{\theta_\infty} \in \big(\bigcup_{i\in I} M_i\big)/\theta_\infty$ with $\mathfrak{c} \in M_i$ we have $h_k(h_{ik}(\mathfrak{c})) = [h_{ik}(\mathfrak{c})]_{\theta_\infty} = [\mathfrak{c}]_{\theta_\infty}$. Hence, h_k is an isomorphism (h_k is compatible with operations since it is a part of the limit cone), $\mathbf{M}_k \cong \lim \mathbf{M}_i$. In other words, the direct limit is trivial for finite $\langle I, \leq\rangle$.

Theorem 3.13. *Let* $\{\mathbf{M}_i \,|\, i \in I\}$ *be a weak direct family. Then the limit cone of* $\{\mathbf{M}_i \,|\, i \in I\}$ *satisfies* DLP *with respect to* $\{\mathbf{M}_i \,|\, i \in I\}$.

Proof. Let $\{h_i \colon \mathbf{M}_i \to \lim \mathbf{M}_i \,|\, i \in I\}$ be the limit cone of $\{\mathbf{M}_i \,|\, i \in I\}$. Take a family $\{g_i \colon \mathbf{M}_i \to \mathbf{N} \,|\, i \in I\}$ of morphisms such that $g_i = h_{ij} \circ g_j$ ($i \leq j$). We check the existence and uniqueness of a morphism $h \colon \lim \mathbf{M}_i \to \mathbf{N}$, where $g_i = h_i \circ h$ ($i \in I$).

First, each $\mathfrak{a} \in \bigcup_{i\in I} M_i$ belongs to some M_i. Hence, define $h \colon \big(\bigcup_{i\in I} M_i\big)/\theta_\infty \to N$ by

$$h\big([\mathfrak{a}]_{\theta_\infty}\big) = g_i(\mathfrak{a}), \qquad \text{where } \mathfrak{a} \in M_i. \tag{10}$$

For every $\mathfrak{a} \in M_i$ and $\mathfrak{b} \in M_j$ we have

$$\begin{aligned}[]
[\mathfrak{a}]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} [\mathfrak{b}]_{\theta_\infty} &= \theta_\infty(\mathfrak{a}, \mathfrak{b}) = \textstyle\bigvee_{k\geq i,j} h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) \leq \\
&\leq \textstyle\bigvee_{k\geq i,j} g_k(h_{ik}(\mathfrak{a})) \approx^{\mathbf{N}} g_k(h_{jk}(\mathfrak{b})) = \bigvee_{k\geq i,j} g_i(\mathfrak{a}) \approx^{\mathbf{N}} g_j(\mathfrak{b}) = g_i(\mathfrak{a}) \approx^{\mathbf{N}} g_j(\mathfrak{b}).
\end{aligned}$$

Thus, $[\mathfrak{a}]_{\theta_\infty} = [\mathfrak{b}]_{\theta_\infty}$ implies $g_i(\mathfrak{a}) = g_j(\mathfrak{b})$. As a consequence, h defined by (10) is a well-defined $\approx$-morphism. Now for any n-ary $f \in F$, and $[\mathfrak{a}_1]_{\theta_\infty}, \ldots, [\mathfrak{a}_n]_{\theta_\infty} \in \big(\bigcup_{i\in I} M_i\big)/\theta_\infty$ there are indices $i_1, \ldots, i_n, l \in I$ such that $i_1, \ldots, i_n \leq l$, and $\mathfrak{a}_1 \in M_{i_1}, \ldots, \mathfrak{a}_n \in M_{i_n}$. Using Lemma 3.7, it follows that

$$\begin{aligned}
h\big(f^{\lim \mathbf{M}_i}([\mathfrak{a}_1]_{\theta_\infty}, \ldots, [\mathfrak{a}_n]_{\theta_\infty})\big) &= h\big(\big[f^{\mathbf{M}_l}\big(h_{i_1 l}(\mathfrak{a}_1), \ldots, h_{i_n l}(\mathfrak{a}_n)\big)\big]_{\theta_\infty}\big) = \\
&= g_l\big(f^{\mathbf{M}_l}(h_{i_1 l}(\mathfrak{a}_1), \ldots, h_{i_n l}(\mathfrak{a}_n))\big) = f^{\mathbf{N}}\big(g_l(h_{i_1 l}(\mathfrak{a}_1)), \ldots, g_l(h_{i_n l}(\mathfrak{a}_n))\big) = \\
&= f^{\mathbf{N}}\big(g_{i_1}(\mathfrak{a}_1), \ldots, g_{i_n}(\mathfrak{a}_n)\big) = f^{\mathbf{N}}\big(h([\mathfrak{a}_1]_{\theta_\infty}), \ldots, h([\mathfrak{a}_n]_{\theta_\infty})\big).
\end{aligned}$$

That is, $h \colon \lim \mathbf{M}_i \to \mathbf{N}$ is the required morphism with $g_i(\mathfrak{a}) = h([\mathfrak{a}]_{\theta_\infty}) = h(h_i(\mathfrak{a})) = (h_i \circ h)(\mathfrak{a})$.

Finally, let $h' \colon \lim \mathbf{M}_i \to \mathbf{N}$ be a morphism satisfying $g_i = h_i \circ h'$. Every $\mathfrak{a} \in \bigcup_{i\in I} M_i$ belongs to M_i for some $i \in I$. That is, $h'\big([\mathfrak{a}]_{\theta_\infty}\big) = h'(h_i(\mathfrak{a})) = g_i(\mathfrak{a}) = h(h_i(\mathfrak{a})) = h\big([\mathfrak{a}]_{\theta_\infty}\big)$, proving the uniqueness. □

Theorem 3.14. *Let* $\{\mathbf{M}_i \,|\, i \in I\}$ *be a weak direct family and let* $\mathbf{N}$ *be an* $\mathbf{L}$*-algebra. Then there is a family* $\{g_i \colon \mathbf{M}_i \to \mathbf{N} \,|\, i \in I\}$ *of morphisms satisfying* DLP *w.r.t.* $\{\mathbf{M}_i \,|\, i \in I\}$ *iff* $\mathbf{N} \cong \lim \mathbf{M}_i$.

Proof. The proof is analogous to that one known from the ordinary case, so we give only a sketch.

"⇒": For $\mathbf{N}$ being $\lim \mathbf{M}_i$, the limit cone $\{h_i \colon \mathbf{M}_i \to \lim \mathbf{M}_i \,|\, i \in I\}$ satisfies DLP w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$. Hence, it suffices to prove that for any families of morphisms $\{g_i \colon \mathbf{M}_i \to \mathbf{N} \,|\, i \in I\}$, $\{g'_i \colon \mathbf{M}_i \to \mathbf{N}' \,|\, i \in I\}$, satisfying DLP w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$ we have $\mathbf{N} \cong \mathbf{N}'$. Since both families satisfy DLP w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$, there are uniquely determined morphisms $g : \mathbf{N} \to \mathbf{N}'$, $g' : \mathbf{N}' \to \mathbf{N}$, where $g_i = g'_i \circ g'$, $g'_i = g_i \circ g$. Consequently $g_i = g'_i \circ g' = (g_i \circ g) \circ g' = g_i \circ (g \circ g')$. Thus, $g \circ g' = \mathrm{id}_N$. Analogously, $g' \circ g = \mathrm{id}_{N'}$. Using the morphism theorems [5], one can conclude that g, g' are mutually inverse isomorphisms between $\mathbf{N}$ and $\mathbf{N}'$. That is, for $\mathbf{N}'$ being $\lim \mathbf{M}_i$ we have $\mathbf{N} \cong \lim \mathbf{M}_i$.

"⇐": For $\mathbf{N} \cong \lim \mathbf{M}_i$ we can take morphisms $g_i \colon \mathbf{M}_i \to \mathbf{N}$ such that $g_i = h_i \circ h$ with $h \colon \lim \mathbf{M}_i \to \mathbf{N}$ being an isomorphism. It is routine to check that $\{g_i \colon \mathbf{M}_i \to \mathbf{N}\}$ satisfies DLP w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$. □

In the ordinary case, any algebra is isomorphic to a direct limit of finitely presented algebras. The following theorem presents an analogous characterization for $\mathbf{L}$-algebras.

Theorem 3.15. *Every* $\mathbf{L}$*-algebra is isomorphic to a direct limit of a direct family of finitely presented* $\mathbf{L}$*-algebras.*

Proof. Let $\mathbf{M}$ be an $\mathbf{L}$-algebra. For brevity, we identify $\mathbf{M}$ with $\mathbf{T}(X)/\theta(R)$, where X is a set of variables and R is a binary $\mathbf{L}$-relation of $T(X)$. Recall that for every $Y \subseteq X$ we can consider a restriction $\theta(R)|\mathbf{T}(Y)$ of $\theta(R)$ on $\mathbf{T}(Y)$. Now let us have the index set

$$I = \big\{\langle Y, S\rangle \,|\, Y \subseteq X,\ Y \text{ is finite, } S \text{ is a finite restriction of } \theta(R)|\mathbf{T}(Y)\big\}.$$

We can define a partial order $\leq$ on I by

$$\langle Y_i, S_i\rangle \leq \langle Y_j, S_j\rangle \quad \text{iff} \quad Y_i \subseteq Y_j \text{ and } S_i \subseteq S_j.$$

Obviously, $\langle I, \leq\rangle$ is directed. For brevity, we denote indices of the form $\langle Y_i, S_i\rangle$, $\langle Y_j, S_j\rangle, \ldots$ simply by $i, j, \ldots$

We introduce morphisms $h_{ij}\colon \mathbf{T}(Y_i)/\theta(S_i) \to \mathbf{T}(Y_j)/\theta(S_j)$ $(i \leq j)$ defined by $h_{ij}\big([t]_{\theta(S_i)}\big) = [t]_{\theta(S_j)}$. Clearly, every h_{ij} satisfies (4) and (5). Thus, $\langle I, \leq\rangle$, $\{\mathbf{T}(Y_i)/\theta(S_i) \,|\, i \in I\}$ together with h_{ij}'s is a weak direct family. Moreover, it is even a direct family. Indeed, take $[t_i]_{\theta(S_i)} \in \mathbf{T}(Y_i)/\theta(S_i)$, $[t_j]_{\theta(S_j)} \in \mathbf{T}(Y_j)/\theta(S_j)$. There is $k \geq i, j$ such that $Y_k = Y_i \cup Y_j$ and $S_k(t_i, t_j) = \theta(R)(t_i, t_j)$. Clearly, for every $l \geq k$ we have

$$\begin{aligned} h_{ik}\big([t_i]_{\theta(S_i)}\big) \approx^{\mathbf{T}(Y_k)/\theta(S_k)} h_{jk}\big([t_j]_{\theta(S_j)}\big) = [t_i]_{\theta(S_k)} \approx^{\mathbf{T}(Y_k)/\theta(S_k)} [t_j]_{\theta(S_k)} = \theta(S_k)(t_i, t_j) = \theta(R)(t_i, t_j) = \\ = \theta(S_l)(t_i, t_j) = [t_i]_{\theta(S_l)} \approx^{\mathbf{T}(Y_l)/\theta(S_l)} [t_j]_{\theta(S_l)} = h_{il}\big([t_i]_{\theta(S_i)}\big) \approx^{\mathbf{T}(Y_l)/\theta(S_l)} h_{jl}\big([t_j]_{\theta(S_j)}\big), \end{aligned}$$

showing that $\langle I, \leq\rangle$, $\{\mathbf{T}(Y_i)/\theta(S_i) \,|\, i \in I\}$ together with h_{ij}'s is a direct family.

The proof is finished by showing that there is a family $\{h_i\colon \mathbf{T}(Y_i)/\theta(S_i) \to \mathbf{T}(X)/\theta(R) \,|\, i \in I\}$ of morphisms satisfying DLP w.r.t. $\{\mathbf{T}(Y_i)/\theta(S_i) \,|\, i \in I\}$. Then $\mathbf{T}(X)/\theta(R) \cong \lim \mathbf{T}(Y_i)/\theta(S_i)$ on account of Theorem 3.14. In the rest of the proof, we denote by $\mathrm{var}(t)$ the set of all variables occurring in the term t.

Put $h_i\big([t]_{\theta(S_i)}\big) = [t]_{\theta(R)}$ for every $t \in T(Y_i)$. Each h_i is a morphism, and $h_i = h_{ij} \circ h_j$ $(i \leq j)$. Take a family $\{g_i\colon \mathbf{T}(Y_i)/\theta(S_i) \to \mathbf{N} \,|\, i \in I\}$ of morphisms with $g_i = h_{ij} \circ g_j$ $(i \leq j)$, and let us define $h\colon \mathbf{T}(X)/\theta(R) \to \mathbf{N}$ by

$$h\big([t]_{\theta(R)}\big) = g_i\big([t]_{\theta(S_i)}\big),$$

where $t \in T(X)$, and $i \in I$ such that $\mathrm{var}(t) \subseteq Y_i$. Note that $h\big([t]_{\theta(R)}\big)$ does not depend on the choice of $i \in I$ since for $i, j \in I$ with $\mathrm{var}(t) \subseteq Y_i, Y_j$ we can take $l \geq i, j$, and then

$$g_i\big([t]_{\theta(S_i)}\big) \approx^{\mathbf{N}} g_j\big([t]_{\theta(S_j)}\big) = g_l\big(h_{il}\big([t]_{\theta(S_i)}\big)\big) \approx^{\mathbf{N}} g_l\big(h_{jl}\big([t]_{\theta(S_j)}\big)\big) = g_l\big([t]_{\theta(S_l)}\big) \approx^{\mathbf{N}} g_l\big([t]_{\theta(S_l)}\big) = 1$$

yields $g_i\big([t]_{\theta(S_i)}\big) = g_j\big([t]_{\theta(S_j)}\big)$. Now take $t_i, t_j \in T(X)$ with $\mathrm{var}(t_i) \subseteq Y_i$, and $\mathrm{var}(t_j) \subseteq Y_j$. For $k \in I$, $k \geq i, j$ such that $\theta(R)(t_i, t_j) = \theta(S_k)(t_i, t_j)$, it follows that

$$\begin{aligned} [t_i]_{\theta(R)} \approx^{\mathbf{T}(X)/\theta(R)} [t_j]_{\theta(R)} = \theta(R)(t_i, t_j) = \theta(S_k)(t_i, t_j) = [t_i]_{\theta(S_k)} \approx^{\mathbf{T}(Y_k)/\theta(S_k)} [t_j]_{\theta(S_k)} \leq \\ \leq g_k\big([t_i]_{\theta(S_k)}\big) \approx^{\mathbf{N}} g_k\big([t_j]_{\theta(S_k)}\big) = g_k\big(h_{ik}\big([t_i]_{\theta(S_i)}\big)\big) \approx^{\mathbf{N}} g_k\big(h_{jk}\big([t_j]_{\theta(S_j)}\big)\big) = g_i\big([t_i]_{\theta(S_i)}\big) \approx^{\mathbf{N}} g_j\big([t_j]_{\theta(S_j)}\big). \end{aligned}$$

Thus, $[t_i]_{\theta(R)} = [t_j]_{\theta(R)}$ implies $g_i\big([t_i]_{\theta(S_i)}\big) = g_j\big([t_j]_{\theta(S_j)}\big)$. Altogehter, h is a well-defined $\approx$-morphism. For any n-ary $f \in F$, and $t_1, \ldots, t_n \in T(X)$, there is $k \in I$ such that $\mathrm{var}(t_i) \subseteq Y_k$ $(i = 1, \ldots, n)$. We have

$$\begin{aligned} h\big(f^{\mathbf{T}(X)/\theta(R)}([t_1]_{\theta(R)}, \ldots, [t_n]_{\theta(R)})\big) = h\big([f(t_1, \ldots, t_n)]_{\theta(R)}\big) = \\ = g_k\big([f(t_1, \ldots, t_n)]_{\theta(S_k)}\big) = g_k\big(f^{\mathbf{T}(Y_k)/\theta(S_k)}([t_1]_{\theta(S_k)}, \ldots, [t_n]_{\theta(S_k)})\big) = \\ = f^{\mathbf{N}}\big(g_k\big([t_1]_{\theta(S_k)}\big), \ldots, g_k\big([t_1]_{\theta(S_k)}\big)\big) = f^{\mathbf{N}}\big(h\big([t_1]_{\theta(R)}\big), \ldots, h\big([t_n]_{\theta(R)}\big)\big). \end{aligned}$$

Hence, h is a morphism. In addition to that, $g_i\big([t]_{\theta(S_i)}\big) = h\big([t]_{\theta(R)}\big) = h\big(h_i\big([t]_{\theta(S_i)}\big)\big)$, for every $t \in T(Y_i)$, i.e. $g_i = h_i \circ h$. Finally, we check the uniqueness of h. Let $h'\colon \mathbf{T}(X)/\theta(R) \to \mathbf{N}$ be a morphism satisfying $g_i = h_i \circ h'$ $(i \in I)$. It is immediate that for $t \in T(X)$, $\mathrm{var}(t) \subseteq Y_i$ we have $h'\big([t]_{\theta(R)}\big) = h'\big(h_i\big([t]_{\theta(S_i)}\big)\big) = g_i\big([t]_{\theta(S_i)}\big) = h\big(h_i\big([t]_{\theta(S_i)}\big)\big) = h\big([t]_{\theta(R)}\big)$. Altogether, $h\colon \mathbf{T}(X)/\theta(R) \to \mathbf{N}$ is a unique morphism with $g_i = h_i \circ h$ $(i \leq j)$. □

Remark 3.16. Using the generalization of direct unions of $\mathbf{L}$-algebras [5], one can show (by standard arguments) that every weak direct limit is isomorphic to a direct union of a directed family of $\mathbf{L}$-algebras.

Theorem 3.17. *Let $\{\mathbf{M}_i \mid\, \in I\}$ be a direct family and let $\{h_i\colon \mathbf{M}_i \to \lim \mathbf{M}_i \,|\, i \in I\}$ be the limit cone of $\lim \mathbf{M}_i$. Suppose $h\colon \mathbf{N} \to \lim \mathbf{M}_i$ is a morphism, where $\mathbf{N}$ is a finitely presented $\mathbf{L}$-algebra. Then there exists $k \in I$ and a morphism $g\colon \mathbf{N} \to \mathbf{M}_k$ such that $h = g \circ h_k$.*

Proof. Since $\mathbf{N}$ is supposed to be finitely presented, we can identify $\mathbf{N}$ with some $\mathbf{T}(X)/\theta(R)$, where X and R are finite. Thus, let us assume a morphism $h\colon \mathbf{T}(X)/\theta(R) \to \lim \mathbf{M}_i$ is given. It is obvious that $\mathbf{T}(X)/\theta(R)$ is generated [5] by $\big\{[x]_{\theta(R)} \,|\, x \in X\big\}$. For every variable $x \in X$ there is an index $i_x \in I$ such that $h\big([x]_{\theta(R)}\big) \in h_{i_x}(M_{i_x})$. Since there are only finitely many variables in X, we can choose $j \in I$ with $j \geq i_x$ $(x \in X)$. Clearly,

$$h\big([x]_{\theta(R)}\big) \in h_{i_x}(M_{i_x}) = h_j(h_{i_x j}(M_{i_x})) \subseteq h_j(M_j)$$

for each $x \in X$. Therefore, $h(h_{\theta(R)}(X)) \subseteq h_j(M_j)$, where $h_{\theta(R)}\colon \mathbf{T}(X) \to \mathbf{T}(X)/\theta(R)$ is a natural morphism. Following this observation, for each $x \in X$ there is $\mathfrak{a}_x \in M_j$ such that $h\big([x]_{\theta(R)}\big) = h_j(\mathfrak{a}_x) \in h_j(M_j)$. Hence, we introduce a mapping $v\colon X \to M_j$ by putting $v(x) = \mathfrak{a}_x$ $(x \in X)$. By definition, $h_j(v(x)) = h\big([x]_{\theta(R)}\big)$ for each $x \in X$. Since for $v^\sharp$ we have $h_j(v^\sharp(t)) = h\big([t]_{\theta(R)}\big)$ for all $t \in T(X)$, it follows that $v^\sharp \circ h_j = h_{\theta(R)} \circ h$.

Recall that R is finite, i.e. $\mathrm{Supp}(R) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_m, t'_m\rangle\}$. Since $\{\mathbf{M}_i \mid \in I\}$ is a direct family, (6) yields that for each $i = 1, \ldots, m$ there is $k_i \in I$ such that $\theta_\infty(v^\sharp(t_i), v^\sharp(t'_i)) = h_{jk_i}(v^\sharp(t_i)) \approx^{\mathbf{M}_{k_i}} h_{jk_i}(v^\sharp(t'_i))$, see Remark 3.6. Following this observation, for $k \geq k_1, \ldots, k_m$ we have

$$\theta(R)(t_i, t'_i) = [t_i]_{\theta(R)} \approx^{\mathbf{T}(X)/\theta(R)} [t'_i]_{\theta(R)} \leq h\big([t_i]_{\theta(R)}\big) \approx^{\lim \mathbf{M}_i} h\big([t'_i]_{\theta(R)}\big) = h_j(v^\sharp(t_i)) \approx^{\lim \mathbf{M}_i} h_j(v^\sharp(t'_i)) =$$
$$= \theta_\infty(v^\sharp(t_i), v^\sharp(t'_i)) = h_{jk_i}(v^\sharp(t_i)) \approx^{\mathbf{M}_{k_i}} h_{jk_i}(v^\sharp(t'_i)) \leq h_{jk}(v^\sharp(t_i)) \approx^{\mathbf{M}_k} h_{jk}(v^\sharp(t'_i)).$$

Thus, $R(t_i, t'_i) \leq h_{jk}(v^\sharp(t_i)) \approx^{\mathbf{M}_k} h_{jk}(v^\sharp(t'_i)) = \theta_{v^\sharp \circ h_{jk}}(t_i, t'_i)$ for each $i = 1, \ldots, m$. Since $\theta_{v^\sharp \circ h_{jk}}$ is a congruence and $\theta(R)$ is generated by R, we get $\theta(R) \subseteq \theta_{v^\sharp \circ h_{jk}}$. Finally, put $g\big([t]_{\theta(R)}\big) = h_{jk}(v^\sharp(t))$. For $t, t' \in T(X)$ we have

$$[t]_{\theta(R)} \approx^{\mathbf{T}(X)/\theta(R)} [t']_{\theta(R)} = \theta(R)(t, t') \leq \theta_{v^\sharp \circ h_{jk}}(t, t') = h_{jk}(v^\sharp(t)) \approx^{\mathbf{M}_k} h_{jk}(v^\sharp(t')) = g\big([t]_{\theta(R)}\big) \approx^{\mathbf{M}_k} g\big([t']_{\theta(R)}\big),$$

i.e. g is a well-defined $\approx$-morphism. For any n-ary $f \in F$, and $[t_1]_{\theta(R)}, \ldots, [t_n]_{\theta(R)} \in T(X)/\theta(R)$ we have

$$g\big(f^{\mathbf{T}(X)/\theta(R)}\big([t_1]_{\theta(R)}, \ldots, [t_n]_{\theta(R)}\big)\big) = g\big([f(t_1, \ldots, t_n)]_{\theta(R)}\big) = h_{jk}\big(v^\sharp\big(f(t_1, \ldots, t_n)\big)\big) =$$
$$= h_{jk}\big(f^{\mathbf{M}_j}\big(v^\sharp(t_1), \ldots, v^\sharp(t_n)\big)\big) = f^{\mathbf{M}_k}\big(h_{jk}(v^\sharp(t_1)), \ldots, h_{jk}(v^\sharp(t_n))\big) = f^{\mathbf{M}_k}\big(g([t_1]_{\theta(R)}), \ldots, g([t_n]_{\theta(R)})\big).$$

Hence, $g\colon \mathbf{T}(X)/\theta(R) \to \mathbf{M}_k$ is a morphism. In addition to that,

$$h\big([t]_{\theta(R)}\big) = h_j(v^\sharp(t)) = h_k(h_{jk}(v^\sharp(t))) = h_k\big(g\big([t]_{\theta(R)}\big)\big) = (g \circ h_k)\big([t]_{\theta(R)}\big)$$

holds for all $[t]_{\theta(R)} \in T(X)/\theta(R)$, i.e. $h = g \circ h_k$. □

Remark 3.18. The existence of the morphism given by Theorem 3.17 is limited to direct families. In the bivalent case, every weak direct family is a direct family, thus Theorem 3.17 coincides with the well-known image factorization theorem for the ordinary algebras. The following example illustrates that postulating (6) is necessary when $\mathbf{L}$ is a general complete residuated lattice.

Example 3.19. Take $\mathbf{L} = [0, 1]$. Let us have a family $\{\mathbf{M}_i \mid i \in \mathbb{N}\}$ of $\mathbf{L}$-algebras $\mathbf{M}_i = \big\langle M_i, \approx^{\mathbf{M}_i}, \emptyset\big\rangle$ such that $M_i = \{\mathfrak{a}_i, \mathfrak{b}_i\}$, and $\mathfrak{a}_i \approx^{\mathbf{M}_i} \mathfrak{b}_i = \frac{i}{2i+1}$. That is, $\mathfrak{a}_1 \approx^{\mathbf{M}_1} \mathfrak{b}_1 = \frac{1}{3}$, $\mathfrak{a}_2 \approx^{\mathbf{M}_2} \mathfrak{b}_2 = \frac{2}{5}$, $\mathfrak{a}_3 \approx^{\mathbf{M}_3} \mathfrak{b}_3 = \frac{3}{7}, \ldots$. Clearly, $\langle \mathbb{N}, \leq\rangle$ is a directed index set, the universe sets M_i $(i \in \mathbb{N})$ are pairwise disjoint, and $\{h_{ij}\colon \mathbf{M}_i \to \mathbf{M}_j \mid i \leq j\}$, where $h_{ij}(\mathfrak{a}_i) = \mathfrak{a}_j$, and $h_{ij}(\mathfrak{b}_i) = \mathfrak{b}_j$ is a family of morphisms satisfying (4) and (5). Altogether, $\langle \mathbb{N}, \leq\rangle$ with $\{\mathbf{M}_i \mid i \in \mathbb{N}\}$, and $\{h_{ij}\colon \mathbf{M}_i \to \mathbf{M}_j \mid i \leq j\}$ is a weak direct family. On the other hand, it is not a direct family, because for $i, k \in \mathbb{N}$ with $i \leq k$ we have $h_{ik}(\mathfrak{a}_i) \approx^{\mathbf{M}_k} h_{ik}(\mathfrak{b}_i) < h_{i,k+1}(\mathfrak{a}_i) \approx^{\mathbf{M}_{k+1}} h_{i,k+1}(\mathfrak{b}_i)$. Moreover, we have

$$\theta_\infty(\mathfrak{a}_i, \mathfrak{b}_i) = \textstyle\bigvee_{k \geq i} h_{ik}(\mathfrak{a}_i) \approx^{\mathbf{M}_k} h_{ik}(\mathfrak{b}_i) = \frac{1}{2},$$

i.e. $\big(\bigcup_{i \in \mathbb{N}} M_i\big)/\theta_\infty = \big\{[\mathfrak{a}_i]_{\theta_\infty}, [\mathfrak{b}_i]_{\theta_\infty}\big\}$. Since $\lim \mathbf{M}_i$ is of the empty type ($F^{\lim \mathbf{M}_i} = \emptyset$), it readily follows that $T(X) = X$. Thus, for $X = \{x, y\}$, and $R \in \mathbf{L}^{T(X) \times T(X)}$, where $R(x, y) = R(y, x) = \frac{1}{2}$, and $R(x, x) = R(y, y) = 1$, we have $\theta(R) = R$. Therefore, $\mathbf{T}(X)/\theta(R)$ is finitely presented. Now let $h\colon \mathbf{T}(X)/\theta(R) \to \lim \mathbf{M}_i$ be defined by $h\big([x]_{\theta(R)}\big) = [\mathfrak{a}_j]_{\theta_\infty}$, $h\big([y]_{\theta(R)}\big) = [\mathfrak{b}_j]_{\theta_\infty}$. Clearly, h is an $\approx$-morphism and thus a morphism. Suppose $h = g \circ h_k$, where $g : T(X)/\theta(R) \to M_k$ and $h_k : \mathbf{M}_k \to \lim \mathbf{M}_i$ is a morphism of the limit cone of $\lim \mathbf{M}_i$. In such a case, $h = g \circ h_k$ yields $g\big([x]_{\theta(R)}\big) = \mathfrak{a}_k$, $g\big([y]_{\theta(R)}\big) = \mathfrak{b}_k$. Thus, g cannot be an $\approx$-morphism since $\frac{1}{2} \not\leq \frac{k}{2k+1}$.

Let us stress a consequential property of the generalized direct limits. If $\mathbf{L}$ is infinite and $\{\mathbf{M}_i \mid i \in I\}$ is a weak direct family which is not a direct family, there can be elements $\mathfrak{a} \in M_i$, $\mathfrak{b} \in M_j$ the homomorphic images of which are distinct in every $\mathbf{M}_k$ for $k \geq i, j$. However, it can happen that $\theta_\infty(\mathfrak{a}, \mathfrak{b}) = 1$, i.e. $[\mathfrak{a}]_{\theta_\infty} = [\mathfrak{b}]_{\theta_\infty}$ due to the general suprema used in (9). Such a situation is apparently ill at least from the standpoint of compatibility with the ordinary direct limits. Indeed, the skeleton $\mathrm{ske}(\lim \mathbf{M}_i)$ (i.e. an ordinary algebra being the functional part of $\lim \mathbf{M}_i$) is then not isomorphic to the ordinary direct limit of skeletons $\mathrm{ske}(\mathbf{M}_i)$. On the other hand, such a situation cannot occur for direct families of $\mathbf{L}$-algebras.

Theorem 3.20. *Let $\{\mathbf{M}_i \mid i \in I\}$ be a direct family. Then* $\mathrm{ske}(\lim \mathbf{M}_i) \cong \lim\{\mathrm{ske}(\mathbf{M}_i) \mid i \in I\}$.

Proof. The claim is almost evident. It suffices to show that for $\mathfrak{a} \in M_i$, and $\mathfrak{b} \in M_j$ we have $\theta_\infty(\mathfrak{a}, \mathfrak{b}) = 1$ iff there exists $k \geq i, j$ such that $h_{ik}(\mathfrak{a}) = h_{jk}(\mathfrak{b})$ since then $\big(\bigcup_{i \in I} M_i\big)/\theta_\infty$ coincides with its classical counterpart. So, assume that $\theta_\infty(\mathfrak{a}, \mathfrak{b}) = 1$. Since $\{\mathbf{M}_i \mid i \in I\}$ is a direct family, there is some index $k \geq i, j$ such that $h_{ik}(\mathfrak{a}) \approx^{\mathbf{M}_k} h_{jk}(\mathfrak{b}) = 1$. That is, $h_{ik}(\mathfrak{a}) = h_{jk}(\mathfrak{b})$. The converse implication holds trivially. Altogether, ske($\lim \mathbf{M}_i$) $\cong$ $\lim$ ske($\mathbf{M}_i$) since the corresponding functions on ske($\lim \mathbf{M}_i$) and $\lim$ ske($\mathbf{M}_i$) are defined the same way. □

Example 3.21. Let us consider $\mathbf{L} = [0,1]$ as the structure of truth degrees. We can take a weak direct family from (b) of Remark 3.2. It is evident that $\lim \mathbf{M}_i$ is a trivial $\mathbf{L}$-algebra but there is not any $j \in I$ such that $h_{ij}(\mathfrak{a}_i) = h_{ij}(\mathfrak{b}_i)$. On the other hand, $\lim$ ske($\mathbf{M}_i$) is a two-element (ordinary) algebra. Hence, Theorem 3.20 is not true for general weak direct families of $\mathbf{L}$-algebras.

Remark 3.22. (a) Let us mention an alternative way to generalize direct limits. In the ordinary case [1, 22], a direct limit is sometimes defined to be a factorization of a special subalgebra of a direct product. The direct limit of $\mathbf{L}$-algebras can be approached analogously. Recall that we have already generalized all the necessary notions [5]. Namely, for a directed index set $\langle I, \leq\rangle$, and a family $\{h_{ij}\colon \mathbf{M}_i \to \mathbf{M}_j \,|\, i \leq j\}$ of morphisms satisfying (4) and (5) we can define a set $M^{\triangleleft}$ by

$$M^{\triangleleft} = \big\{\mathfrak{a} \in \textstyle\prod_{i\in I} M_i \,|\, \text{there is } i \in I \text{ such that for } j,k \in I \text{ with } i \leq j \leq k \text{ we have } h_{jk}(\mathfrak{a}(j)) = \mathfrak{a}(k)\big\}.$$

Described verbally, $M^{\triangleleft}$ represents a subset of $\prod_{i\in I} M_i$ every element of which respects h_{ij}'s. Furthermore, we define a binary $\mathbf{L}$-relation $\theta^{\triangleleft}$ on $M^{\triangleleft}$ by $\theta^{\triangleleft}(\mathfrak{a},\mathfrak{b}) = \bigvee_{i\in I} \bigwedge_{k\geq i} \mathfrak{a}(k) \approx^{\mathbf{M}_k} \mathfrak{b}(k)$ for every $\mathfrak{a},\mathfrak{b} \in M^{\triangleleft}$. It can be shown that (i) $\emptyset \neq M^{\triangleleft}$ is a subuniverse of $\prod_{i\in I} \mathbf{M}_i$, and $\theta^{\triangleleft}$ is a congruence; (ii) for every weak direct family $\{\mathbf{M}_i \,|\, i \in I\}$ we have $\lim \mathbf{M}_i \cong \mathbf{M}^{\triangleleft}/\theta^{\triangleleft}$; (iii) for every directed index set $\langle I, \leq\rangle$ and a family $\{h_{ij}\colon \mathbf{M}_i \to \mathbf{M}_j \,|\, i \leq j\}$ of morphisms satisfying (4) and (5) there is a weak direct family $\{g_{ij}\colon \mathbf{N}_i \to \mathbf{N}_j \,|\, i \leq j\}$ such that $\lim \mathbf{N}_i \cong \mathbf{M}^{\triangleleft}/\theta^{\triangleleft}$. The proof is left to the reader.

(b) Direct limits in context of particular structures associated with fuzzy sets were studied in [16]. The paper describes direct limits of join spaces which are associated with direct families of fuzzy sets. Hence, [16] deals with classical direct limits of classical structures which are somehow related to collections of fuzzy sets.

4. Reduced products

We define reduced products of $\mathbf{L}$-algebras by means of previously defined constructions [5] in much the same way as in the ordinary case. Later on, we introduce a special property called safeness and show its relationship to the essential property (6) of direct families. The key issue of generalizing reduced products to fuzzy setting is how to define a suitable congruence relation on $\prod_{i\in I} \mathbf{M}_i$ with respect to a given filter F over I. Recall that in the classical case we put

$$\langle \mathfrak{a},\mathfrak{b}\rangle \in \theta_F \quad \text{iff} \quad \{i \in I \,|\, \mathfrak{a}(i) = \mathfrak{b}(i)\} \in F.$$

Thus, on the verbal level: "$\langle \mathfrak{a},\mathfrak{b}\rangle \in \theta_F$ iff the set of indices on which $\mathfrak{a}$ equals $\mathfrak{b}$ is large (i.e. belongs to a filter F)." In what follows, we will proceed in two steps. First, we try to generalize the notion of "being equal on indices from $X \in F$". Then, using such a graded equality with respect to some index set, we define an $\mathbf{L}$-relation representing for every $\mathfrak{a},\mathfrak{b} \in \prod_{i\in I} M_i$ a degree to which $\mathfrak{a}$ equals $\mathfrak{b}$ over a large set of indices.

In the sequel, we use an ordinary filter. That is, we do not fuzzify the notion of a filter itself. We denote a filter by F, and the elements of F will be denoted $X, Y, Z, \ldots$ (there is no danger of confusion with the symbol of a type of an $\mathbf{L}$-algebra and with sets of variables, because we use a fixed type and we do not use variables and terms anymore). The rest of this section is devoted to reduced products. Ultraproducts are not discussed.

Definition 4.1. Let $\{\mathbf{M}_i \,|\, i \in I\}$ be a family of $\mathbf{L}$-algebras and let F be a filter over I. Then for every $\mathfrak{a},\mathfrak{b} \in \prod_{i\in I} M_i$ and $X \in F$ we define the truth degree $[\![\mathfrak{a} \approx \mathfrak{b}]\!]_X \in L$ by

$$[\![\mathfrak{a} \approx \mathfrak{b}]\!]_X = \textstyle\bigwedge_{i\in X} \mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i).$$

Lemma 4.2. *Let $\{\mathbf{M}_i \,|\, i \in I\}$ be a family of $\mathbf{L}$-algebras and let F be a filter over I. Then*

(i) *for every $X, Y \in F$, such that $X \subseteq Y$ we have $[\![\mathfrak{a} \approx \mathfrak{b}]\!]_Y \leq [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X$,*
(ii) $\mathfrak{a} \approx^{\prod_{i\in I}\mathbf{M}_i} \mathfrak{b} \leq [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X$ *for every $X \in F$,*
(iii) $\bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X = \bigvee_{X_1,\ldots,X_n\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_{X_1\cap\cdots\cap X_n}$.

Proof. (i) follows directly by Definition 4.1.

(ii): Since $X \subseteq I \in F$, (i) yields $\mathfrak{a} \approx^{\prod_{i\in I}\mathbf{M}_i} \mathfrak{b} = [\![\mathfrak{a} \approx \mathfrak{b}]\!]_I \leq [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X$.

(iii): The "$\leq$"-part follows easily since for each $X \in F$ we have $X = X \cap \cdots \cap X$. Conversely, if $X_1,\ldots,X_n \in F$ then $X_1 \cap \cdots \cap X_n \in F$ since every filter is closed under finite intersections. Hence, the "$\geq$"-part is also evident. □

Now we use $[\![\mathfrak{a} \approx \mathfrak{b}]\!]_X$ to define a suitable $\mathbf{L}$-relation on $\prod_{i\in I} M_i$.

Definition 4.3. Let $\{\mathbf{M}_i \mid i \in I\}$ be a family of $\mathbf{L}$-algebras and let F be a filter over I. We define the binary $\mathbf{L}$-relation θ_F on $\prod_{i\in I} M_i$ by

$$\theta_F(\mathfrak{a}, \mathfrak{b}) = \textstyle\bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X$$

for all $\mathfrak{a}, \mathfrak{b} \in \prod_{i\in I} M_i$.

Remark 4.4. On the verbal level, $[\![\mathfrak{a} \approx \mathfrak{b}]\!]_X$ expresses the truth degree to which it is true that $\mathfrak{a}$ is equal to $\mathfrak{b}$ over all indices taken from X. Since $X \in F$ are thought of as large subsets, $\theta_F(\mathfrak{a}, \mathfrak{b})$ can be understood as the degree to which "there is a large X such that $\mathfrak{a}$ equals $\mathfrak{b}$ over all indices from X".

Lemma 4.5. *Let $\{\mathbf{M}_i \mid i \in I\}$ be a family of $\mathbf{L}$-algebras and let F be a filter over I. Then θ_F is a congruence.*

Proof. By Lemma 4.2 it readily follows that $\approx^{\prod_{i\in I}\mathbf{M}_i} \subseteq \theta_F$. Moreover, reflexivity and symmetry of θ_F follow directly by reflexivity and symmetry of every $\approx^{\mathbf{M}_i}$, respectively. Thus, it suffices to check transitivity and compatibility with functions of $\prod_{i\in I}\mathbf{M}_i$. Using Lemma 4.2, for $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \prod_{i\in I} M_i$ we have

$$\begin{aligned}
\theta_F(\mathfrak{a}, \mathfrak{b}) \otimes \theta_F(\mathfrak{b}, \mathfrak{c}) &= \textstyle\bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X \otimes \bigvee_{Y\in F} [\![\mathfrak{b} \approx \mathfrak{c}]\!]_Y = \bigvee_{X,Y\in F} \big([\![\mathfrak{a} \approx \mathfrak{b}]\!]_X \otimes [\![\mathfrak{b} \approx \mathfrak{c}]\!]_Y\big) = \\
&= \textstyle\bigvee_{X,Y\in F} \big(\bigwedge_{i\in X} \mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i) \otimes \bigwedge_{j\in Y} \mathfrak{b}(j) \approx^{\mathbf{M}_j} \mathfrak{c}(j)\big) \leq \\
&\leq \textstyle\bigvee_{X,Y\in F} \bigwedge_{i,j\in X\cap Y} \big(\mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i) \otimes \mathfrak{b}(j) \approx^{\mathbf{M}_j} \mathfrak{c}(j)\big) \leq \\
&\leq \textstyle\bigvee_{X,Y\in F} \bigwedge_{i\in X\cap Y} \big(\mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i) \otimes \mathfrak{b}(i) \approx^{\mathbf{M}_i} \mathfrak{c}(i)\big) \leq \\
&\leq \textstyle\bigvee_{X,Y\in F} \bigwedge_{i\in X\cap Y} \mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{c}(i) = \bigvee_{X,Y\in F} [\![\mathfrak{a} \approx \mathfrak{c}]\!]_{X\cap Y} = \bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{c}]\!]_X = \theta_F(\mathfrak{a}, \mathfrak{c}).
\end{aligned}$$

Thus, θ_F is transitive. Now we check the compatibility. Take an n-ary $f^{\prod_{i\in I}\mathbf{M}_i}$, and $\mathfrak{a}_1, \mathfrak{b}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}_n \in \prod_{i\in I} M_i$. Applying the compatibility of all $\approx^{\mathbf{M}_i}$'s, we have

$$\begin{aligned}
&\theta_F(\mathfrak{a}_1, \mathfrak{b}_1) \otimes \cdots \otimes \theta_F(\mathfrak{a}_n, \mathfrak{b}_n) = \textstyle\bigvee_{X_1,\ldots,X_n\in F} \bigotimes_{i=1}^{n} \bigwedge_{j_i\in X_i} \mathfrak{a}(j_i) \approx^{\mathbf{M}_{j_i}} \mathfrak{b}(j_i) \leq \\
&\quad\leq \textstyle\bigvee_{X_1,\ldots,X_n\in F} [\![f^{\prod_{i\in I}\mathbf{M}_i}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \approx f^{\prod_{i\in I}\mathbf{M}_i}(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)]\!]_{X_1\cap\cdots\cap X_n} = \\
&\quad= \textstyle\bigvee_{X\in F} [\![f^{\prod_{i\in I}\mathbf{M}_i}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \approx f^{\prod_{i\in I}\mathbf{M}_i}(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)]\!]_X = \theta_F\big(f^{\prod_{i\in I}\mathbf{M}_i}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n), f^{\prod_{i\in I}\mathbf{M}_i}(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)\big).
\end{aligned}$$

Altogether, θ_F is a congruence on $\prod_{i\in I}\mathbf{M}_i$. □

Finally, we introduce the reduced product of $\mathbf{L}$-algebras.

Definition 4.6. Let $\{\mathbf{M}_i \mid i \in I\}$ be a family of $\mathbf{L}$-algebras and let F be a filter over I. Then $\big(\prod_{i\in I}\mathbf{M}_i\big)/\theta_F$ denoted by $\prod_F \mathbf{M}_i$ is called the *reduced product of* $\{\mathbf{M}_i \mid i \in I\}$ *modulo* F.

Remark 4.7. Clearly, θ_F and the corresponding $\prod_F \mathbf{M}_i$ are determined by $\{\mathbf{M}_i \mid i \in I\}$ and the filter F over I. In borderline cases, θ_F behaves the same way as in the ordinary case. Indeed, when F is an improper filter (i.e. $\emptyset \in F$), we have $\theta_F(\mathfrak{a}, \mathfrak{b}) = 1$ for all $\mathfrak{a}, \mathfrak{b} \in \prod_{i\in I} M_i$. Thus, $\prod_F \mathbf{M}_i$ is a trivial (one-element) $\mathbf{L}$-algebra. If F is a trivial filter (i.e. F is a proper filter and $\{i_0\} \in F$ for $i_0 \in I$) it follows that $\theta_F = \approx^{\mathbf{M}_{i_0}}$ for certain $i_0 \in I$. That is, $\prod_F \mathbf{M}_i \cong \mathbf{M}_{i_0}$. Finally, if $F = \{I\}$ then clearly $\theta_F = \approx^{\prod_{i\in I}\mathbf{M}_i}$, i.e. $\prod_F \mathbf{M}_i \cong \prod_{i\in I}\mathbf{M}_i$.

In the ordinary case, the reduced product $\prod_F \mathbf{M}_i$ is isomorphic to a special direct limit. In the subsequent development, we present an analogous characterization for fuzzy setting. The reduced product of $\mathbf{L}$-algebras will be characterized as a direct limit of certain weak direct family of $\mathbf{L}$-algebras.

Let $\{\mathbf{M}_i \mid i \in I\}$ be a family of $\mathbf{L}$-algebras and let us have a filter F over I. For every $X \in F$ we can consider the direct product $\prod_{i\in X}\mathbf{M}_i$. For brevity, let $\mathbf{M}_X$ denote $\prod_{i\in X}\mathbf{M}_i$. That is, $M_X = \prod_{i\in X} M_i$, $\approx^{\mathbf{M}_X} = \approx^{\prod_{i\in X}\mathbf{M}_i}$, for an n-ary function symbol f let $f^{\mathbf{M}_X}$ denote $f^{\prod_{i\in X}\mathbf{M}_i}$. It readily follows that

$$\mathfrak{a} \approx^{\mathbf{M}_X} \mathfrak{b} = \textstyle\bigwedge_{i\in X} \mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i) = [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X.$$

In addition to that, F can be partially ordered using the ordinary set inclusion. Namely, $\langle F, \supseteq\rangle$ can be thought of as a (downward) directed index set. Clearly, M_X, M_Y are disjoint for each distinct $X, Y \in F$. For $X \supseteq Y$ we define a morphism $h_{XY} : \mathbf{M}_X \to \mathbf{M}_Y$ by

$$h_{XY}(\mathfrak{a})(i) = \mathfrak{a}(i)$$

for every $\mathfrak{a} \in M_X$ and $i \in Y$. It is easily seen that $\{h_{XY} : \mathbf{M}_X \to \mathbf{M}_Y \mid X \supseteq Y\}$ satisfies conditions (4) and (5). As a consequence, $\langle F, \supseteq\rangle$, $\{\mathbf{M}_X \mid X \in F\}$, and morphisms h_{XY} $(X \supseteq Y)$ form a weak direct family.

In the sequel, we use the following technical lemma.

Lemma 4.8. *Let* $h\colon \mathbf{M} \to \mathbf{N}$ *be a morphism and let* ϕ *be a congruence on* $\mathbf{M}$ *such that* $\phi \subseteq \theta_h$. *Then* $h = h_\phi \circ g$, *where* $g\colon \mathbf{M}/\phi \to \mathbf{N}$ *is a uniquely determined morphism.*

Proof. The assertion follows by morphism theorems [5] using analogous arguments as in the ordinary case. □

Theorem 4.9. *Let* $\{\mathbf{M}_i \,|\, i \in I\}$ *be a family of* $\mathbf{L}$*-algebras and let* F *be a filter over* I. *Then* $\prod_F \mathbf{M}_i \cong \lim \mathbf{M}_X$.

Proof. We present a family $\{h_X : \mathbf{M}_X \to \prod_F \mathbf{M}_i\}$ of morphisms satisfying DLP with respect to $\{\mathbf{M}_X \,|\, X \in F\}$. Recall that $I \in F$, and $\mathbf{M}_I$ stands for $\prod_{i\in I}\mathbf{M}_i$. Thus, $h_{IX} : \prod_{i\in I}\mathbf{M}_i \to \mathbf{M}_X$ $(X \in F)$ are surjective morphisms. For $\mathfrak{a},\mathfrak{b} \in \prod_{i\in I} M_i$ we have

$$\begin{aligned}\theta_{h_{IX}}(\mathfrak{a},\mathfrak{b}) = h_{IX}(\mathfrak{a}) \approx^{\mathbf{M}_X} h_{IX}(\mathfrak{b}) = \textstyle\bigwedge_{i\in X} h_{IX}(\mathfrak{a})(i) \approx^{\mathbf{M}_i} h_{IX}(\mathfrak{b})(i) = \\ = \textstyle\bigwedge_{i\in X} \mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i) = [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X \leq \theta_F(\mathfrak{a},\mathfrak{b}).\end{aligned}$$

Therefore, $\theta_{h_{IX}} \subseteq \theta_F$ $(X \in F)$. Let $h_{\theta_F} : \prod_{i\in I}\mathbf{M}_i \to \prod_F \mathbf{M}_i$ denote the natural morphism. By Lemma 4.8, for every $X \in F$ there is a uniquely determined morphism $h_X : \mathbf{M}_X \to \prod_F \mathbf{M}_i$ satisfying $h_{\theta_F} = h_{IX} \circ h_X$. Moreover, we can apply (5) to obtain $h_{\theta_F} = h_{IY} \circ h_Y = h_{IX} \circ h_{XY} \circ h_Y$, i.e. $h_{IX} \circ h_{XY} \circ h_Y = h_{IX} \circ h_X$. Thus, the surjectivity of h_{IX} yields $h_X = h_{XY} \circ h_Y$.

Let us have a family $\{g_X : \mathbf{M}_X \to \mathbf{N} \,|\, X \in F\}$ of morphisms satisfying $g_X = h_{XY} \circ g_Y$ $(X \supseteq Y)$. It remains to show that there is a uniquely determined morphism $g\colon \prod_F \mathbf{M}_i \to \mathbf{N}$ such that $g_X = h_X \circ g$ $(X \in F)$. First, for $\mathfrak{a},\mathfrak{b} \in \prod_{i\in I} M_i$ it follows that

$$\begin{aligned}\theta_F(\mathfrak{a},\mathfrak{b}) &= \textstyle\bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X = \bigvee_{X\in F} \bigwedge_{i\in X} \mathfrak{a}(i) \approx^{\mathbf{M}_i} \mathfrak{b}(i) = \\ &= \textstyle\bigvee_{X\in F} \bigwedge_{i\in X} h_{IX}(\mathfrak{a})(i) \approx^{\mathbf{M}_i} h_{IX}(\mathfrak{b})(i) = \bigvee_{X\in F} h_{IX}(\mathfrak{a}) \approx^{\mathbf{M}_X} h_{IX}(\mathfrak{b}) \leq \\ &\leq \textstyle\bigvee_{X\in F} g_X(h_{IX}(\mathfrak{a})) \approx^{\mathbf{N}} g_X(h_{IX}(\mathfrak{b})) = \bigvee_{X\in F} g_I(\mathfrak{a}) \approx^{\mathbf{N}} g_I(\mathfrak{b}) = g_I(\mathfrak{a}) \approx^{\mathbf{N}} g_I(\mathfrak{b}) = \theta_{g_I}(\mathfrak{a},\mathfrak{b}).\end{aligned}$$

Hence, $\theta_F \subseteq \theta_{g_I}$. By Lemma 4.8, there is a uniquely determined morphism $g\colon \prod_F \mathbf{M}_i \to \mathbf{N}$, where $g_I = h_{\theta_F} \circ g$. Finally, $g_I = h_{\theta_F} \circ g = h_{IX} \circ h_X \circ g$ and $g_I = h_{IX} \circ g_X$ by the assumption. As a consequence, $g_X = h_X \circ g$ due to the surjectivity of h_{IX}. Thus, $\{h_X : \mathbf{M}_X \to \prod_F \mathbf{M}_i\}$ satisfies DLP w.r.t. $\{\mathbf{M}_X \,|\, X \in F\}$. Now $\prod_F \mathbf{M}_i \cong \lim \mathbf{M}_X$ is a consequence of Theorem 3.14. □

As we have seen, for $\{\mathbf{M}_i \,|\, i \in I\}$ and a filter F over I, $\{\mathbf{M}_X \,|\, X \in F\}$ is a weak direct family of $\mathbf{L}$-algebras. Suppose $\{\mathbf{M}_X \,|\, X \in F\}$ is a direct family of $\mathbf{L}$-algebras and let us look whether the essential property (6) has a natural translation in terms of the properties of F.

Definition 4.10. Let $\{\mathbf{M}_i \,|\, i \in I\}$ be a family of $\mathbf{L}$-algebras. A filter F over I is called *safe with respect to* $\{\mathbf{M}_i \,|\, i \in I\}$ if for every $\mathfrak{a},\mathfrak{b} \in \prod_{i\in I} M_i$ there is $X \in F$ such that $\theta_F(\mathfrak{a},\mathfrak{b}) = [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X$. If $\mathcal{K}$ is a class of $\mathbf{L}$-algebras and F is safe w.r.t. every (I-indexed) family of $\mathbf{L}$-algebras taken from $\mathcal{K}$ then F is called $\mathcal{K}$*-safe.* If F is $\mathcal{K}$-safe for arbitrary class $\mathcal{K}$ of $\mathbf{L}$-algebras then F is said to be *safe.* If F is safe with respect to a family $\{\mathbf{M}_i \,|\, i \in I\}$ then $\prod_F \mathbf{M}_i$ is called the *safe reduced product of* $\{\mathbf{M}_i \,|\, i \in I\}$ *modulo* F.

Remark 4.11. Safeness of a filter F with respect to $\{\mathbf{M}_i \,|\, i \in I\}$ is a nontrivial property.

(a) If $F = \{I\}$ then F is safe. Also every trivial and improper filter is safe.

(b) If $\theta_F(\mathfrak{a},\mathfrak{b})$ is compact for all $\mathfrak{a},\mathfrak{b} \in \prod_{i\in I} M_i$ then F is safe w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$. Indeed, for any $\mathfrak{a},\mathfrak{b} \in \prod_{i\in I} M_i$ there are $X_1,\dots,X_n \in F$ such that $\theta_F(\mathfrak{a},\mathfrak{b}) = \bigvee_{i=1}^n [\![\mathfrak{a} \approx \mathfrak{b}]\!]_{X_i}$. Since $X_1 \cap \dots \cap X_n \in F$, it follows that $\theta_F(\mathfrak{a},\mathfrak{b}) \leq [\![\mathfrak{a} \approx \mathfrak{b}]\!]_{X_1\cap\dots\cap X_n}$. The converse inequality holds trivially. Thus, if each $a \in L$ is compact (see [7]) then every filter F is safe.

(c) Take $\mathbf{L} = [0,1]$ as the structure of truth degrees. Let us have an index set $\mathbb{N}$ and a family $\{\mathbf{M}_i \,|\, i \in \mathbb{N}\}$ of $\mathbf{L}$-algebras of the empty type, where $M_i = \{\mathfrak{a},\mathfrak{b}\}$ and $\mathfrak{a} \approx^{\mathbf{M}_i} \mathfrak{b} = 1 - \frac{1}{i}$ $(i \in \mathbb{N})$. Thus, $\mathfrak{a} \approx^{\mathbf{M}_1} \mathfrak{b} = 0$, $\mathfrak{a} \approx^{\mathbf{M}_2} \mathfrak{b} = \frac{1}{2}$, $\mathfrak{a} \approx^{\mathbf{M}_3} \mathfrak{b} = \frac{2}{3}$, etc. Let F be Fréchet filter over $\mathbb{N}$. Take $\mathfrak{a}',\mathfrak{b}' \in \prod_{i\in\mathbb{N}} M_i$, where $\mathfrak{a}'(i) = \mathfrak{a}$ and $\mathfrak{b}'(i) = \mathfrak{b}$ for all $i \in \mathbb{N}$. Clearly, we have $\theta_F(\mathfrak{a}',\mathfrak{b}') = 1$ but $[\![\mathfrak{a}' \approx \mathfrak{b}']\!]_X < 1$ for every $X \in F$. Hence, F is not safe w.r.t. $\{\mathbf{M}_i \,|\, i \in \mathbb{N}\}$.

Lemma 4.12. *Let* $\{\mathbf{M}_i \,|\, i \in I\}$ *be a family of* $\mathbf{L}$*-algebras and let* F *be a filter over* I. *Then for* $\mathfrak{a} \in M_X$, $\mathfrak{b} \in M_Y$ *and any* $Z \in F$ *such that* $X, Y \supseteq Z$ *we have*

$$h_{XZ}(\mathfrak{a}) \approx^{\mathbf{M}_Z} h_{YZ}(\mathfrak{b}) = [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_Z,$$

where $\mathfrak{a}',\mathfrak{b}' \in \prod_{i\in I} M_i$ *satisfy* $h_{IX}(\mathfrak{a}') = \mathfrak{a}$ *and* $h_{IY}(\mathfrak{b}') = \mathfrak{b}$.

Proof. Clearly, we have

$$h_{XZ}(\mathfrak{a}) \approx^{\mathbf{M}_Z} h_{YZ}(\mathfrak{b}) = \bigwedge_{i\in Z} h_{XZ}(\mathfrak{a})(i) \approx^{\mathbf{M}_i} h_{YZ}(\mathfrak{b})(i) = \bigwedge_{i\in Z} h_{XZ}(h_{IX}(\mathfrak{a}'))(i) \approx^{\mathbf{M}_i} h_{YZ}(h_{IY}(\mathfrak{b}'))(i) = \\ = \bigwedge_{i\in Z} h_{IZ}(\mathfrak{a}')(i) \approx^{\mathbf{M}_i} h_{IZ}(\mathfrak{b}')(i) = \bigwedge_{i\in Z} \mathfrak{a}'(i) \approx^{\mathbf{M}_i} \mathfrak{b}'(i) = [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_Z,$$

which is the desired equality. □

Theorem 4.13. *Filter F is safe w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$ iff $\{\mathbf{M}_X \,|\, X \in F\}$ is a direct family.*

Proof. "⇒": Let F be safe w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$. Take $\mathfrak{a} \in M_X$, $\mathfrak{b} \in M_Y$. We have to show that there is $Z \in F$ such that $X, Y \supseteq Z$ and

$$h_{XZ}(\mathfrak{a}) \approx^{\mathbf{M}_Z} h_{YZ}(\mathfrak{b}) = h_{XZ'}(\mathfrak{a}) \approx^{\mathbf{M}_{Z'}} h_{YZ'}(\mathfrak{b})$$

for every $Z' \in F$ with $Z \supseteq Z'$. Let us have $\mathfrak{a}', \mathfrak{b}' \in \prod_{i\in I} M_i$ such that $h_{IX}(\mathfrak{a}') = \mathfrak{a}$ and $h_{IY}(\mathfrak{b}') = \mathfrak{b}$. Since F is safe, we have $\theta_F(\mathfrak{a}', \mathfrak{b}') = [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_{Z_0}$ for certain $Z_0 \in F$. Put $Z = Z_0 \cap X \cap Y$. Clearly, for every $Z' \in F$ such that $Z \supseteq Z'$, it follows that $\theta_F(\mathfrak{a}', \mathfrak{b}') = [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_{Z_0} = [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_Z \geq [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_{Z'}$. Moreover, Lemma 4.2 (i) yields $[\![\mathfrak{a}' \approx \mathfrak{b}']\!]_Z \leq [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_{Z'}$. Altogether, using Lemma 4.12, we obtain

$$h_{XZ}(\mathfrak{a}) \approx^{\mathbf{M}_Z} h_{YZ}(\mathfrak{b}) = [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_Z = [\![\mathfrak{a}' \approx \mathfrak{b}']\!]_{Z'} = h_{XZ'}(\mathfrak{a}) \approx^{\mathbf{M}_{Z'}} h_{YZ'}(\mathfrak{b}).$$

Hence, $\{\mathbf{M}_X \,|\, X \in F\}$ is a direct family.

"⇐": Let $\{\mathbf{M}_X \,|\, X \in F\}$ be a direct family. For $\mathfrak{a}, \mathfrak{b} \in \prod_{i\in I} M_i$ there is some $Z \in F$ such that

$$[\![\mathfrak{a} \approx \mathfrak{b}]\!]_Z = h_{IZ}(\mathfrak{a}) \approx^{\mathbf{M}_Z} h_{IZ}(\mathfrak{b}) = h_{IZ'}(\mathfrak{a}) \approx^{\mathbf{M}_{Z'}} h_{IZ'}(\mathfrak{b}) = [\![\mathfrak{a} \approx \mathfrak{b}]\!]_{Z'}$$

holds for every $Z' \in F$, $Z \supseteq Z'$. Thus, we have

$$\theta_F(\mathfrak{a}, \mathfrak{b}) = \bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_X \leq \bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_{X\cap Z} = \bigvee_{X\in F} [\![\mathfrak{a} \approx \mathfrak{b}]\!]_Z = [\![\mathfrak{a} \approx \mathfrak{b}]\!]_Z.$$

The converse inequality follows by the definition of θ_F. That is, filter F is safe w.r.t. $\{\mathbf{M}_i \,|\, i \in I\}$. □

Remark 4.14. (a) The construction of a safe reduced product is compatible with its ordinary counterpart in the sense of preserving skeletons:

$$\mathrm{ske}(\textstyle\prod_F \mathbf{M}_i) \cong \mathrm{ske}(\lim \mathbf{M}_X) \cong \lim\{\mathrm{ske}(\mathbf{M}_X) \,|\, X \in F\} \cong \prod_F \mathrm{ske}(\mathbf{M}_i).$$

This is an immediate consequence of Theorem 3.20, Theorem 4.9, and Theorem 4.13. On the contrary, one can use Remark 4.11 to observe that $\mathrm{ske}(\prod_F \mathbf{M}_i) \cong \prod_F \mathrm{ske}(\mathbf{M}_i)$ does not hold for general reduced products.

(b) In [6] we used fuzzy sets of generalized implications between identities, so-called Horn clauses with truth-weighted premises, to characterize sur-reflective classes and semivarieties of **L**-algebras. In [4], R. Bělohlávek used fuzzy sets of identities to characterize varieties of **L**-algebras. All these results pass for any complete residuated lattice as a structure of truth degrees. In case of quasivarieties, however, finiteness of **L** was used to ensure that each weak direct family is a direct family, and each filter is safe. We also showed that it is not possible to work with unrestricted weak direct families and arbitrary filters because Horn classes of **L**-algebras are not closed under arbitrary direct limits and reduced products in general. Even if we develop quasivarieties with safe reduced products without invoking a connection to direct limits, Theorem 4.9 and Theorem 4.13 show that the notion of safeness corresponds to the essential property (6) of direct families. An open problem is whether there are other reasonable generalizations of reduced products and direct limits which lead to characterization of quasivarieties over wider subclasses of residuated lattices (hints can be found in [19]).

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