Extended fuzzy equational logic

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Abstract. We study fuzzy equational logics extended by additional deduction rules. It is shown that equational deduction rules which preserve substitutions can be represented by Horn clauses with truth-weighted premises. This ensures us to reduce provability in extended fuzzy equational logics to provability in fuzzy Horn logic. We work in Pavelka-style (we use truth-evaluated syntax) and prove Pavelka-style completeness of extended fuzzy equational logics.

1 Introduction

Equational logics and their modifications are extensively used in mathematics (universal algebra, model theory) and computer science (formal specification, formal verification, etc.). Equational reasoning is very natural from the user's point of view because "working with equations" is almost a matter of everyday practice. Equational logics have been developed and used at least for two reasons. First, equational logics are particular fragments of predicate logic in which we study entailments (syntactic and semantic) from collections of formulas involving identities (equations) as well as other problems which are difficult to handle in full predicate languages (e.g., definability of classes of structures by sets of formulas). Second, equational reasoning plays an important role in universal algebra because equationally definable classes of algebras (like groups, rings, lattices, etc.) are the most important and studied ones. Thus, equational logic is an important means which allows us to draw conclusions about classes of algebras.

In this paper we study a problem which is connected with equational logic developed in the framework of *fuzzy logic in narrow sense* [11, 13, 15]. Our starting point is *fuzzy equational logic* (FEL) which was introduced in papers [2, 4] of R. Bělohlávek. Fuzzy equational logic is a calculus for reasoning with identities in *graded style* (for the notion of a *graded consequence* see also [8, 9]). This enables us to deal with identities (equations) which are satisfied only to certain degrees. The exploration of partially true statements is by no means artificial. For instance, if a behavior of a complex system cannot be described precisely, we are compelled to confine ourselves to approximate descriptions in which identities satisfied to degrees might be useful. In [6], we presented *fuzzy Horn logic* (FHL) as an extension of fuzzy equational logic which uses more complex formulas than identities, namely particular implications between identities. Both calculi (FEL and FHL) are developed in Pavelka-style [11, 13, 15], i.e. we use a fixed structure of truth degrees, we define a degree of semantic entailment and a degree of provability (syntactic entailment), and completeness theorems say that the provability degree is equal to the degree of semantic entailment. The aim of this paper is to prove completeness of fuzzy equational logics extended by additional deduction rules. By an *extended fuzzy equational logic* we mean fuzzy equational logic as introduced in [2] which uses also additional deduction rules. To prove completeness of extended FEL, we proceed so that the additional rules of extended FEL will be represented by particular formulas of fuzzy Horn logic. We then take advantage of completeness of fuzzy Horn logic to prove completeness of the extended fuzzy equational logic. Note that the problem of extending FEL by additional rules is interesting from several viewpoints which are sort of hidden in the classical case. For instance, unlike the deduction rules of the classical equational logic, the deduction rules of FEL have nontrivial semantic component—the rules operate on truth-weighted formulas. It is then natural to ask whether such complex rules can be represented by (accordingly complex) formulas of fuzzy Horn logic. This topic and related issues are discussed later in the paper.

In Section 2 we briefly summarize basic syntactic and semantic notions of FHL. Section 3 introduces extended FEL and presents the completeness theorem. Finally, in Section 4 we elaborate the proof.

2 Preliminaries

We use complete residuated lattices as the structures of truth degrees. A *complete residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that (i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1, (ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, (iii) $\langle \otimes, \rightarrow\rangle$ is an *adjoint pair,* i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is true for each $a, b, c \in L$ (so called *adjointness property*). Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are interpretations (truth functions of) logical connectives "fuzzy conjunction" and "fuzzy implication". The class of complete residuated lattices includes, among other structures, all finite residuated lattices. An operation ${}^*\colon L \to L$ satisfying (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \to b)^* \leq a^* \to b^*$, is called a *truth stresser* [6, 12]. Truth stressers can be seen as various interpretations (truth functions of) logical connective "very true", see [11, 12]. A complete residuated lattice $\mathbf{L}$ endowed with a truth stresser * will be denoted by $\mathbf{L}^*$. Two extreme cases of truth stressers are (i) identity, i.e. $a^* = a$ $(a \in L)$; (ii) so-called *globalization* [16]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

If $\mathbf{L}$ is a chain, the globalization on $\mathbf{L}$ coincides with the Baaz's $\triangle$-operation on $\mathbf{L}$ [1, 11].

Having $\mathbf{L}$ given as our structure of truth degrees, we define the usual structural notions. An $\mathbf{L}$*-set* A (or fuzzy set with truth degrees in $\mathbf{L}$) in a universe set U is a mapping $A\colon U \to L$, $A(u) \in L$ being interpreted as the truth degree of "element u belongs to A". A *binary* $\mathbf{L}$*-relation* R on U is an $\mathbf{L}$-set in the universe set $U \times U$, i.e. it is a mapping $R\colon U \times U \to L$. $\mathbf{L}^U$ denotes the collection of all $\mathbf{L}$-sets in U. For any $\mathbf{L}$-set $A\colon U \to L$, the *support set of* A, denoted by $\mathrm{Supp}(A)$, is defined by $\mathrm{Supp}(A) = \{u \in U \mid A(u) > 0\}$. An $\mathbf{L}$-set A is called *finite* if $\mathrm{Supp}(A)$ is finite. An $\mathbf{L}$-set A with $\mathrm{Supp}(A) \subseteq \{u\}$, denoted by $\{{}^{A(u)}/u\}$, is called a *singleton.* Basic operations with $\mathbf{L}$-sets are defined componentwise using operations of $\mathbf{L}$. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. For $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ we write $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$; and $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

In the sequel, we present a brief introduction to fuzzy Horn logic. We begin with syntactic notions. A type is a collection F of function symbols together with their arities. An $\mathbf{L}$-language of fuzzy Horn logic of type F consists of (at least denumerable) set X of variables, function symbols $f \in F$, a binary relation symbol $\approx$ standing for (fuzzy) equality, a set $\{\overline{a} \mid a \in L\}$ of symbols of truth degrees, and symbols of logical connectives $\Rightarrow$ (implication) and $\curlywedge$ (conjunction). For brevity and since there is no danger of confusion, we identify each $\overline{a}$ with $a \in L$, i.e. we identify each symbol of truth degree with the truth degree itself. From now on, we tacitly assume we are given an $\mathbf{L}$-language of type F. Terms $t, s, \ldots$ are defined as usual; $T(X)$ stands for the collection of all terms. Identities are denoted by $t \approx t'$, $s \approx s'$, etc. A *Horn clause* (*with truth-weighted premises*), denoted by $P \Rightarrow (t \approx t')$, is a an expresion of the form

$$\langle t_1 \approx t'_1, P(t_1, t'_1)\rangle \curlywedge \cdots \curlywedge \langle t_n \approx t'_n, P(t_n, t'_n)\rangle \Rightarrow (t \approx t'),$$

where P, called $\mathbf{L}$*-set of premises*, is a finite binary $\mathbf{L}$-relation on $T(X)$ with $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$, and $t, t' \in T(X)$. Horn clauses, being the basic formulas of FHL, are the only formulas we take into our considerations (in particular, symbols for truth degrees and identities are not formulas of FHL). Note that $P(s, s') \in L$ can be interpreted as the degree (weight) to which the identity $s \approx s'$ belongs to P. The intended meaning of the Horn clause $P \Rightarrow (t \approx t')$ with $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$ is "if t_1 equals t'_1 in degree (at least) $P(t_1, t'_1)$ and $\cdots$ and t_n equals t'_n in degree (at least) $P(t_n, t'_n)$, then t equals t'". The concept of Horn clauses with truth-weighted premises originated in our investigation of implicationally definable classes of fuzzy structures [6, 7].

Now we are going to introduce the notion of a provability degree, i.e. a degree of syntactic entailment. The main difference between our approach and the classical Horn logic is that we work with truth-weighted formulas and infer "formulas in degrees". This influences, e.g., the notions of a deduction rule and a proof.

Given a Horn clause $P \Rightarrow (t \approx t')$ and a truth degree $a \in L$, the couple $\langle P \Rightarrow (t \approx t'), a\rangle$ is called a *weighted Horn clause.* For brevity, we write $\langle t \approx t', a\rangle$ instead of $\langle \emptyset \Rightarrow (t \approx t'), a\rangle$. $\mathbf{L}$-sets of Horn clauses are denoted by $\Gamma, \Sigma, \ldots$; *Fml* denotes the set of all Horn clauses. Any partial mapping $R\colon (\mathit{Fml} \times L)^n \to \mathit{Fml} \times L$ is called an $\mathbf{L}^*$-deduction rule. Nullary $\mathbf{L}^*$-deduction rules will be called *axioms.* For convenience, instead of

$$R(\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle) = \langle \varphi, a\rangle,$$

we occasionally write

$$R\colon\ \frac{\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle}{\langle \varphi, a\rangle}.$$

Described verbally, $R(\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle) = \langle \varphi, a\rangle$ reads: "from φ_1 in degree a_1 and $\cdots$ and φ_n in degree a_n infer φ in degree a". A system $\mathcal{R}$ of $\mathbf{L}^*$-deduction rules is called an $\mathbf{L}^*$-deductive system.

Let Σ be an $\mathbf{L}$-set of Horn clauses and let $\mathcal{R}$ be an $\mathbf{L}^*$-deductive system. An ($\mathbf{L}^*$*-weighted*) $\mathcal{R}$*-proof* of $\langle P \Rightarrow (t \approx t'), a\rangle$ from Σ, is a finite sequence of weighted Horn clauses $\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_l, a_l\rangle$, where φ_l is $P \Rightarrow (t \approx t')$, $a_l = a$, and for each $i = 1, \ldots, l$ we either have $a_i = \Sigma(\varphi_i)$ or there is an n-ary $\mathbf{L}^*$-deduction rule $R \in \mathcal{R}$ such that $R(\langle \varphi_{i_1}, a_{i_1}\rangle, \ldots, \langle \varphi_{i_n}, a_{i_n}\rangle) = \langle \varphi_i, a_i\rangle$ for some $i_1, \ldots, i_n < i$. A weighted Horn clause

$\langle P \Rightarrow (t \approx t'), b\rangle$ is said to be *$\mathcal{R}$-provable from* Σ if there is an $\mathcal{R}$-proof of $\langle P \Rightarrow (t \approx t'), b\rangle$ from Σ. We denote this fact by $\Sigma \vdash^{\mathcal{R}} \langle P \Rightarrow (t \approx t'), b\rangle$. A Horn clause $P \Rightarrow (t \approx t')$ is called *$\mathcal{R}$-provable from Σ in degree (at least)* $b \in L$ if $\Sigma \vdash^{\mathcal{R}} \langle P \Rightarrow (t \approx t'), b\rangle$. For any Horn clause $P \Rightarrow (t \approx t')$ we define the *degree* $|P \Rightarrow (t \approx t')|^{\mathcal{R}}_{\Sigma}$ *of $\mathcal{R}$-provability of $P \Rightarrow (t \approx t')$ from* Σ by

$$|P \Rightarrow (t \approx t')|^{\mathcal{R}}_{\Sigma} = \bigvee\{a \in L \,|\, \Sigma \vdash^{\mathcal{R}} \langle P \Rightarrow (t \approx t'), a\rangle\}.$$

In what follows, we use the $\mathbf{L}^*$-deductive system $\mathcal{R}_{\mathrm{AG}}$ [17] which is equivalent to that one presented in [6]. Since $\mathcal{R}_{\mathrm{AG}}$ contains a rule which uses substitutions, we first introduce all the necessary notions. A couple (x/r), where $x \in X$ and $r \in T(X)$, is called a *substitution.* Given $t \in T(X)$ and (x/r), $t(x/r)$ denotes the term resulting from t by substitution of all occurrences of x in t by r. For each finite binary $\mathbf{L}$-relation P on $T(X)$, we define a finite binary $\mathbf{L}$-relation $P(x/r)$ on $T(X)$ by

$$\big(P(x/r)\big)(t, t') = \bigvee \{P(s, s') \,|\, s(x/r) = t \text{ and } s'(x/r) = t'\}$$

for all $t, t' \in T(X)$. Note that $P(x/r)$ can be seen as the $\mathbf{L}$-set of premises resulting from P by substitution of all occurrences of x in the $\mathbf{L}$-set of premises P by r.

$\mathcal{R}_{\mathrm{AG}}$ contains the following $\mathbf{L}^*$-deduction rules:

(ARef): $\langle t \approx t, 1\rangle$,

(ASym): $\langle\langle t \approx t', a\rangle \Rightarrow (t' \approx t), a\rangle$,

(ATra): $\langle\langle t \approx t', a\rangle \curlywedge \langle t' \approx t'', b\rangle \Rightarrow (t \approx t''), a \otimes b\rangle$,

(ARep): $\langle\langle t \approx t', a\rangle \Rightarrow (q \approx q'), a\rangle$,

(Cut): $\dfrac{\langle Q \Rightarrow (s \approx s'), a\rangle, \langle\{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle}{\big\langle Q \cup P \Rightarrow (t \approx t'), b \otimes ((c \vee P(s, s')) \rightarrow a)^*\big\rangle}$,

(Sub): $\dfrac{\langle P \Rightarrow (t \approx t'), a\rangle}{\big\langle P(x/r) \Rightarrow \big(t(x/r) \approx t'(x/r)\big), a\big\rangle}$,

(Sup): $\dfrac{\langle P \Rightarrow (t \approx t'), a\rangle,\ \langle P \Rightarrow (t \approx t'), b\rangle}{\langle P \Rightarrow (t \approx t'), a \vee b\rangle}$,

(Wea): $\dfrac{\langle P \Rightarrow (t \approx t'), a\rangle}{\langle Q \cup P \Rightarrow (t \approx t'), a\rangle}$,

where $x \in X$, $q, r, s, s', t, t', t'' \in T(X)$; q has an occurrence of t as a subterm and q' results from q by substitution of one occurrence of t by t'; $a, b, c \in L$.

Remark. (Cut) is a straightforward generalization of the classical cut rule

$$\frac{Q \Rightarrow (s \approx s'), \{s \approx s'\} \cup P \Rightarrow (t \approx t')}{Q \cup P \Rightarrow (t \approx t')}$$

in which we use weighted Horn clauses with truth-weighted premises instead of classical Horn clauses. Observe that the $\mathbf{L}$-set $\{{}^{c}/\langle s, s'\rangle\} \cup P$ of premises which is used in (Cut) can be understood as the $\mathbf{L}$-set of premises P extended by the identity $s \approx s'$ in degree $c \in L$. Thus, (Cut) says that from weighted Horn clauses $\langle Q \Rightarrow (s \approx s'), a\rangle$ and $\langle\{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle$ one can infer $\big\langle Q \cup P \Rightarrow (t \approx t'), b \otimes ((c \vee P(s, s')) \rightarrow a)^*\big\rangle$. The resulting degree $b \otimes ((c \vee P(s, s')) \rightarrow a)^*$ can be interpreted as the degree to which "$\{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t')$ holds (is provable) and it is very true that if $s \approx s'$ belongs to $\{{}^{c}/\langle s, s'\rangle\} \cup P$ then $Q \Rightarrow (s \approx s')$ holds (is provable)". The meaning of "very true" is determined by the chosen truth stresser *. In the sequel, we are going to use globalization which is the interpretation of "fully true". If * is globalization, (Cut) is equivalent to

$$\frac{\langle Q \Rightarrow (s \approx s'), a\rangle, \langle\{{}^{c}/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle}{\langle Q \cup P \Rightarrow (t \approx t'), b\rangle} \qquad \text{if } c \leq a \text{ and } P(s, s') \leq a.$$

That is, (Cut) is used in a nontrivial way iff $((c \vee P(s, s')) \rightarrow a)^* = 1$, i.e. iff $c \vee P(s, s') \leq a$, which is equivalent to $c \leq a$ and $P(s, s') \leq a$. In this case, the resulting formula is inferred in degree b. If $c \nleq a$ or $P(s, s') \nleq a$, the resulting formula is inferred in degree 0 (not interesting from the provability standpoint). This has the following consequence: from $\langle s \approx s', a\rangle$ and $\langle\langle s \approx s', c\rangle \curlywedge \langle s_1 \approx s'_1, c_1\rangle \curlywedge \cdots \curlywedge \langle s_n \approx s'_n, c_n\rangle \Rightarrow (t \approx t'), b\rangle$, where $c \leq a$, one can infer $\langle\langle s_1 \approx s'_1, c_1\rangle \curlywedge \cdots \curlywedge \langle s_n \approx s'_n, c_n\rangle \Rightarrow (t \approx t'), b\rangle$ by (Cut).

Furthermore, we present basic semantic notions of FHL. In order to introduce the interpretation of Horn clauses and semantic entailment we need suitable semantic structures—we will use particular fuzzy structures called algebras with fuzzy equalities. An *algebra with fuzzy equality* [5] of type F, shortly an $\mathbf{L}$-*algebra*, is a triplet $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$, where $\langle M, F^{\mathbf{M}}\rangle$ is a classical algebra of type F and $\approx^{\mathbf{M}}$ is an $\mathbf{L}$-equality on M (a particular binary $\mathbf{L}$-relation on M) such that each function $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is *compatible* with $\approx^{\mathbf{M}}$. In more detail, $\approx^{\mathbf{M}}$ satisfies:

(i) $a \approx^{\mathbf{M}} b = 1$ iff $a = b$;
(ii) $a \approx^{\mathbf{M}} b = b \approx^{\mathbf{M}} a$;
(iii) $a \approx^{\mathbf{M}} b \otimes b \approx^{\mathbf{M}} c \leq a \approx^{\mathbf{M}} c$;
(iv) $a_1 \approx^{\mathbf{M}} b_1 \otimes \cdots \otimes a_n \approx^{\mathbf{M}} b_n \leq f^{\mathbf{M}}(a_1, \ldots, a_n) \approx^{\mathbf{M}} f^{\mathbf{M}}(b_1, \ldots, b_n)$

for any n-ary $f \in F$, and $a, b, c, a_1, b_1, \ldots, a_n, b_n \in M$. Algebras with $\mathbf{L}$-equalities are natural fuzzy structures equipped with functions that map pairwise similar arguments to similar results provided that the interpretation of "being similar" is given by the $\mathbf{L}$-equality $\approx^{\mathbf{M}}$.

Given an $\mathbf{L}$-algebra $\mathbf{M}$, a valuation is any mapping $v : X \to M$ assigning to each variable $x \in X$ its interpretation $v(x) \in M$. The interpretation $\|t\|_{\mathbf{M},v}$ of a term t in an $\mathbf{L}$-algebra $\mathbf{M}$ under a valuation $v\colon X \to M$ is defined as usual. The degree $\|t \approx t'\|_{\mathbf{M},v}$ to which $t \approx t'$ is true in $\mathbf{M}$ under v is defined by

$$\|t \approx t'\|_{\mathbf{M},v} = \|t\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t'\|_{\mathbf{M},v}.$$

The degree $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ to which $P \Rightarrow (t \approx t')$ is true in $\mathbf{M}$ under v is defined by

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = \left(\textstyle\bigwedge_{s,s' \in T(X)} \left(P(s,s') \to \|s \approx s'\|_{\mathbf{M},v}\right)\right)^{*} \to \|t \approx t'\|_{\mathbf{M},v}.$$

The degree $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ to which $P \Rightarrow (t \approx t')$ is valid in $\mathbf{M}$ is defined using the infimum which ranges over all valuations:

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M}} = \textstyle\bigwedge_{v: X \to M} \|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}.$$

For a class $\mathcal{K}$ of $\mathbf{L}$-algebras we define the degree $\|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$ to which $P \Rightarrow (t \approx t')$ is valid in $\mathcal{K}$ by

$$\|P \Rightarrow (t \approx t')\|_{\mathcal{K}} = \textstyle\bigwedge_{\mathbf{M} \in \mathcal{K}} \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}.$$

Given an $\mathbf{L}$-set Σ of Horn clauses, an $\mathbf{L}$-algebra $\mathbf{M}$ is called a *model of* Σ if $\Sigma(P \Rightarrow (t \approx t')) \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}$ for any Horn clause $P \Rightarrow (t \approx t')$. The class of all models of Σ is denoted $\mathrm{Mod}(\Sigma)$. Finally, the degree $\|P \Rightarrow (t \approx t')\|_{\Sigma}$ to which $P \Rightarrow (t \approx t')$ *semantically follows* from Σ is defined by

$$\|P \Rightarrow (t \approx t')\|_{\Sigma} = \|P \Rightarrow (t \approx t')\|_{\mathrm{Mod}(\Sigma)}.$$

The following theorem is a consequence of the general result on completeness of FHL, see [6].

Theorem (completeness of FHL)**.** *Let* $\mathbf{L}^{*}$ *be a finite residuated lattice with globalization. Then for each* $\mathbf{L}$*-set* Σ *of Horn clauses we have* $|P \Rightarrow (t \approx t')|_{\Sigma}^{\mathcal{R}_{\mathrm{AG}}} = \|P \Rightarrow (t \approx t')\|_{\Sigma}$. ☐

3 Extended fuzzy equational logic

Fuzzy equational logic [2] is a logical calculus for reasoning with identities (equalities) in presence of vagueness. FEL is developed in Pavelka-style. FEL defines degrees of syntactic and semantic entailment. Let us stress that the graded approach to entailment is what makes FEL interesting. If we had considered only the "fully true identities", FEL would be, in a sense, trivial. Observe that an identity $t \approx t'$ is fully true in an $\mathbf{L}$-algebra $\mathbf{M}$ (i.e., $\|t \approx t'\|_{\mathbf{M}} = 1$) iff for any valuation v we have that $1 = \|t \approx t'\|_{\mathbf{M},v} = \|t\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t'\|_{\mathbf{M},v}$ which is true iff for any valuation v, $\|t\|_{\mathbf{M},v}$ equals $\|t'\|_{\mathbf{M},v}$. That is, the fact that $t \approx t'$ is fully true in $\mathbf{M}$ does not depend on the $\mathbf{L}$-equality $\approx^{\mathbf{M}}$ (it depends only on the functional part of $\mathbf{M}$). Since the functional part of any $\mathbf{L}$-algebra is a classical algebra, the semantic entailment based only on "fully true identities" would therefore collapse to classical one.

The aim of this section is to present an approach to extend fuzzy equational logic by additional deduction rules. The way we are going to proceed is motivated by the result of [6], where we showed that the completeness of FEL can be obtained as a particular case of completeness of FHL. The present paper shows that completeness of fuzzy equational logics extended by new deduction rules can also be proven using completeness of FHL. An analogous idea was introduced in classical equational logic [14].

Recall that an identity $t \approx t'$ can be thought of as a Horn clause $\emptyset \Rightarrow (t \approx t')$. This is not surprising because $\|t \approx t'\|_{\mathbf{M},v} = \|\emptyset \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ for any $\mathbf{L}$-algebra $\mathbf{M}$ and any valuation v. Thus, an $\mathbf{L}$-set Σ of identities can be understood as an $\mathbf{L}$-set of Horn clauses of the form $\emptyset \Rightarrow (\cdots)$. Conversely, an $\mathbf{L}$-set Σ of Horn clauses, where $\Sigma(P \Rightarrow (t \approx t')) \neq 0$ implies $P = \emptyset$, can be seen as an $\mathbf{L}$-set of identities. This observation shall be used freely.

Before we begin, let us note that in the sequel we will use several $\mathbf{L}^*$-deductive systems at a time. Namely, the $\mathbf{L}^*$-deductive system $\mathcal{R}_{\mathrm{AG}}$ of fuzzy Horn logic, and various $\mathbf{L}^*$-deductive systems of extended fuzzy equational logics. In order to make a clear distinction between these two types of $\mathbf{L}^*$-deductive systems, $\mathbf{L}^*$-deduction rules which deal only with identities will be called *equational* $\mathbf{L}^*$*-deduction rules*; $\mathbf{L}^*$-deductive systems which consist only of equational $\mathbf{L}^*$-deduction rules will be called *equational* $\mathbf{L}^*$*-deductive systems.*

The basic equational $\mathbf{L}^*$-deductive system of FEL [2], here denoted by $\mathcal{R}\mathrm{eq}$, consists of the rules

$$\text{(ERef)}: \langle t \approx t, 1\rangle, \qquad \text{(ESym)}: \frac{\langle t \approx t', a\rangle}{\langle t' \approx t, a\rangle}, \qquad \text{(ETra)}: \frac{\langle t \approx t', a\rangle,\ \langle t' \approx t'', b\rangle}{\langle t \approx t'', a \otimes b\rangle},$$

$$\text{(ERep)}: \frac{\langle t \approx t', a\rangle}{\langle q \approx q', a\rangle}, \qquad \text{(ESub)}: \frac{\langle t \approx t', a\rangle}{\langle t(x/r) \approx t'(x/r), a\rangle}, \qquad \text{(ESup)}: \frac{\langle t \approx t', a\rangle,\ \langle t \approx t', b\rangle}{\langle t \approx t', a \vee b\rangle},$$

where $x \in X$, $q, t, t', t'' \in T(X)$, $a, b \in L$, term q has an occurrence of t as a subterm and q' is a term resulting from q by substitution of one occurrence of t by t'. Note that FEL as presented in [2] does not use (ESup). Nevertheless, by adding (ESup) to the original deductive system of FEL we do not change the provability degree (this is easy to see: inspect the "$\leq$"-part of the proof of [6, Theorem 14]). We keep (ESup) for technical reasons.

Now consider the following problem. Having given an equational $\mathbf{L}^*$-deductive system $\mathcal{R}$, we wish to describe the degrees of $(\mathcal{R}\mathrm{eq} \cup \mathcal{R})$-provability by means of semantic notions (if possible) and establish a Pavelka-style completeness of fuzzy equational logic extended by rules contained in $\mathcal{R}$. It is obvious that for any $\mathbf{L}$-set Σ of identities we have

$$\|t \approx t'\|_{\Sigma} = |t \approx t'|_{\Sigma}^{\mathcal{R}\mathrm{eq}} \leq |t \approx t'|_{\Sigma}^{\mathcal{R}\mathrm{eq} \cup \mathcal{R}}$$

by completeness of FEL [2, 6]. In what follows we are going to show that if $\mathcal{R}$ consists of rules which "behave like deduction schemes" (we will comment on this later on) then there is a class $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma)$ of $\mathbf{L}$-algebras such that $|t \approx t'|_{\Sigma}^{\mathcal{R}\mathrm{eq} \cup \mathcal{R}} = \|t \approx t'\|_{\mathcal{K}}$. Moreover, we are going to show that $\mathcal{K}$ is exactly the model class of the $\mathbf{L}$-set of Horn clauses which encompasses all identities of Σ and Horn clauses that represent the equational $\mathbf{L}^*$-deduction rules from $\mathcal{R}$.

Let us comment on why we do not focus on extending fuzzy equational logic by arbitrary equational $\mathbf{L}^*$-deductive systems and consider only particular ones. In most logical calculi, deduction rules are treated as *deduction schemes.* That is, if one infers φ from $\varphi_1, \ldots, \varphi_k$ using some deduction rule, we usually assume that one can also infer a particularization of φ from particularizations of $\varphi_1, \ldots, \varphi_k$. If we consider formulas of predicate languages (which is the case of equational logics), by a particularization we mean, typically, a *substitution* (terms denoting objects are substituted for object variables). The concept of "deduction rules as deduction schemes" also applies for the rules of fuzzy equational logic, so it is quite natural to require it for the additional rules too. For instance, if we assume an additional equational $\mathbf{L}^*$-deduction rule R such that $R(\langle f(x) \approx g(y), a\rangle) = \langle x \approx y, b\rangle$, we (intuitively) expect $R(\langle f(f(x)) \approx g(y), a\rangle)$ to be defined. If $R(\langle f(f(x)) \approx g(y), a\rangle)$ were not defined, R would not behave as a deduction scheme. What shall we say about the outcome of $R(\langle f(f(x)) \approx g(y), a\rangle)$? It might be intuitively acceptable to expect $R(\langle f(f(x)) \approx g(y), a\rangle) = \langle f(x) \approx y, b'\rangle$. In addition to that, b' should be greater than or equal to b because $\langle f(x) \approx y, b'\rangle$ is a particularization of $\langle x \approx y, b\rangle$. To sum up, we concentrate on rules which preserve, in a sense, all substitutions. Such rules will be called proper rules. A precise definition follows.

Definition 1 (proper rules)**.** Given an n-ary equational $\mathbf{L}^*$-deduction rule R and a substitution (x/r), R is called (x/r)*-preserving* if for every set $\{\langle t_1 \approx t'_1, a_1\rangle, \ldots, \langle t_n \approx t'_n, a_n\rangle\}$, and $a'_1 \geq a_1, \ldots, a'_n \geq a_n$ we have:

IF $R(\cdots \langle t_i \approx t'_i, a_i\rangle \cdots) = \langle t \approx t', a\rangle$,
THEN $R(\cdots \langle t_i(x/r) \approx t'_i(x/r), a'_i\rangle \cdots) = \langle t(x/r) \approx t'(x/r), a'\rangle$ for $a' \in L$ such that $a' \geq a$.

If R is (x/x)-preserving for all $x \in X$ then R is called *nondecreasing.* If R is (x/r)-preserving for all substitutions (x/r) then R is called *proper.*

In order to use fuzzy Horn logic to prove completeness of extended fuzzy equational logic, we need to represent the additional deduction rules of FEL by formulas of FHL. From this point of view, it is essential to point out the relationship between equational $\mathbf{L}^*$-deduction rules and weighted Horn clauses with truth-weighted premises. The intended meaning of an equational $\mathbf{L}^*$-deduction rule

$$\frac{\langle t_1 \approx t'_1, a_1\rangle, \ldots, \langle t_n \approx t'_n, a_n\rangle}{\langle t \approx t', a\rangle}. \tag{2}$$

is "from $t_1 \approx t'_1$ in degree a_1, and $\cdots$ and $t_n \approx t'_n$ in degree a_n infer $t \approx t'$ in degree a". Now consider the following weighted Horn clause corresponding to (2):

$$\langle\langle t_1 \approx t'_1, a_1\rangle \curlywedge \cdots \curlywedge \langle t_n \approx t'_n, a_n\rangle \Rightarrow (t \approx t'), a\rangle. \tag{3}$$

If * is globalization then $\|\langle t_1 \approx t'_1, a_1\rangle \curlywedge \cdots \curlywedge \langle t_n \approx t'_n, a_n\rangle \Rightarrow (t \approx t')\|_{\mathbf{M}} \geq a$ iff for every valuation v we have:

$$\text{IF } \left(\bigwedge_{i=1}^{n} \left(a_i \rightarrow \|t_i \approx t'_i\|_{\mathbf{M},v}\right)\right)^* = 1, \text{ THEN } a \leq \|t \approx t'\|_{\mathbf{M},v}.$$

which is true iff

$$\text{IF } a_i \leq \|t_i \approx t'_i\|_{\mathbf{M},v} \text{ for each } i = 1, \ldots, n, \text{ THEN } a \leq \|t \approx t'\|_{\mathbf{M},v}.$$

Thus, one can see a resemblance between the intended meaning of an equational $\mathbf{L}^*$-deduction rule and the interpretation of the corresponding Horn clause provided that the interpretation of Horn clauses is determined by globalization. According to this observation, we shall represent an equational $\mathbf{L}^*$-deductive system $\mathcal{R}$ by a suitable $\mathbf{L}$-set $\Phi(\mathcal{R})$ of Horn clauses. $\Phi(\mathcal{R})$ is introduced in the following definition.

Definition 2. For each $n \in \mathbb{N}_0$, let Π_n denote the set of all permutations of $\{1, \ldots, n\}$. Let $P \Rightarrow (t \approx t')$ be a Horn clause with $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$. We define the degree to which $P \Rightarrow (t \approx t')$ belongs to $\Phi(\mathcal{R})$ using the supremum over all possible applications of n-ary rules from $\mathcal{R}$:

$$\big(\Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t')) = \bigvee\big\{a \in L \,|\, R \in \mathcal{R},\ R \text{ is } n\text{-ary},\ \pi \in \Pi_n, \text{ and } R^\pi(P) = \langle t \approx t', a\rangle\big\}, \tag{4}$$

where $R^\pi(P)$ is an abbreviation for

$$R\big(\langle t_{\pi(1)} \approx t'_{\pi(1)}, P(t_{\pi(1)}, t'_{\pi(1)})\rangle, \ldots, \langle t_{\pi(n)} \approx t'_{\pi(n)}, P(t_{\pi(n)}, t'_{\pi(n)})\rangle\big). \tag{5}$$

The following assertion is the main result of this paper and will be proven in Section 4.

Theorem (completeness of extended fuzzy equational logic)**.** *Let $\mathbf{L}^*$ be a finite residuated lattice with globalization. Then for every equational $\mathbf{L}^*$-deductive system $\mathcal{R}$ of proper $\mathbf{L}^*$-deduction rules we have*

$$|t \approx t'|_{\Sigma}^{\mathcal{R}\mathrm{eq}\,\cup\,\mathcal{R}} = \|t \approx t'\|_{\mathrm{Mod}(\Sigma\,\cup\,\Phi(\mathcal{R}))}, \tag{6}$$

for every $\mathbf{L}$-set of identities Σ, and all $t, t' \in T(X)$. □

Remark. The restriction on proper rules is not necessary to prove the completeness of extended fuzzy equational logic in general. However, we can show that certain $\mathbf{L}^*$-deduction rules which are not proper cannot be added to $\mathcal{R}$eq without violating (6).

(1) Consider the rule R such that $R(\langle x \approx x, 1\rangle) = \langle x \approx x, 1\rangle$ for a fixed variable $x \in X$; $R(\cdots)$ is undefined otherwise. Clearly, R is not proper because it is not (x/y)-preserving for any $y \neq x$. Still, by adding R to $\mathcal{R}$eq we do not change the provability degree because R is an instance of (ESym). Therefore, fuzzy equational logic extended by R is Pavelka-style complete in sense of (6).

(2) Take R such that $R(\langle f(x) \approx g(y), 1\rangle) = \langle x \approx y, 1\rangle$ (f, g are unary function symbols) for some fixed $x \neq y$; $R(\cdots)$ is undefined otherwise. Obviously, R is not proper. Put $\mathcal{R} = \{R\}$. We show that (6) is violated. For an $\mathbf{L}$-set Σ of identities, where $\Sigma(f(c) \approx g(d)) = 1$ (c, d are nullary function symbols, i.e. object constants), $\Sigma(\cdots) = 0$ otherwise, we have $\|c \approx d\|_{\mathrm{Mod}(\Sigma\,\cup\,\Phi(\mathcal{R}))} = 1$ because

1: $\langle\langle f(x) \approx g(y), 1\rangle \Rightarrow (x \approx y), 1\rangle$,	follows from $\Phi(\mathcal{R})$
2: $\langle\langle f(c) \approx g(y), 1\rangle \Rightarrow (c \approx y), 1\rangle$,	by (Sub) on 1
3: $\langle\langle f(c) \approx g(d), 1\rangle \Rightarrow (c \approx d), 1\rangle$,	by (Sub) on 2
4: $\langle f(c) \approx g(d), 1\rangle$,	follows from Σ
5: $\langle c \approx d, 1\rangle$,	by (Cut) on $3, 4$

is an $\mathcal{R}_{\mathrm{AG}}$-proof of $\langle c \approx d, 1\rangle$ from $\Sigma \cup \Phi(\mathcal{R})$. On the other hand, we clearly have that $|c \approx d|_{\Sigma}^{\mathcal{R}\mathrm{eq}} = \|c \approx d\|_{\Sigma} \neq 1$. Furthermore, there is also a model $\mathbf{M}$ of Σ such that $\|f(x) \approx g(y)\|_{\mathbf{M}} \neq 1$. As a consequence, $|f(x) \approx g(y)|_{\Sigma}^{\mathcal{R}\mathrm{eq}} = \|f(x) \approx g(y)\|_{\Sigma} \neq 1$. Hence, if $\Sigma \vdash^{\mathcal{R}\mathrm{eq}} \langle f(x) \approx g(y), a\rangle$ then $a \neq 1$. This gives that R cannot be used to infer $\langle x \approx y, 1\rangle$ from $\langle f(x) \approx g(y), 1\rangle$ in any $(\mathcal{R}\mathrm{eq} \cup \mathcal{R})$-proof from Σ, i.e. $|c \approx d|_{\Sigma}^{\mathcal{R}\mathrm{eq}\,\cup\,\mathcal{R}} = |c \approx d|_{\Sigma}^{\mathcal{R}\mathrm{eq}} \neq 1$. Thus, (6) is violated.

4 Proof of completeness

In this section, we let $\mathbf{L}^*$ denote a finite residuated lattice with globalization, Σ denotes an $\mathbf{L}$-set of identities, and $\mathcal{R}$ stands for an equational $\mathbf{L}^*$-deductive system which consists of proper rules. Our goal is to prove

$$|t \approx t'|^{\mathcal{R}\mathrm{eq}\,\cup\,\mathcal{R}}_{\Sigma} = |t \approx t'|^{\mathcal{R}_{\mathrm{AG}}}_{\Sigma\,\cup\,\Phi(\mathcal{R})} \tag{7}$$

since then the completeness of the fuzzy equational logic extended by $\mathcal{R}$ would be a consequence of the completeness of fuzzy Horn logic.

Remark. (1) (ERef), (ESym), (ETra), (ERep), and (ESup) are proper rules.

(2) (ESub) is nondecreasing, but it is not proper. For instance, let us have a type $F = \{p\}$ with p being a unary function symbol. For brevity, let R be the substitution rule (in fact, (ESub) encompasses infinitely many rules [6]), which derives $\langle t(x/p(y)) \approx t'(x/p(y)), a\rangle$ from $\langle t \approx t', a\rangle$. We have $R(\langle x \approx y, 1\rangle) = \langle p(y) \approx y, 1\rangle$. Now it is easily seen that R is not (x/y)-preserving since $R(\langle y \approx y, 1\rangle) = \langle y \approx y, 1\rangle \neq \langle p(y) \approx y, 1\rangle$.

The following assertion shows the "$\leq$"-part of (7).

Lemma 3. $|t \approx t'|^{\mathcal{R}\mathrm{eq}\,\cup\,\mathcal{R}}_{\Sigma} \leq |t \approx t'|^{\mathcal{R}_{\mathrm{AG}}}_{\Sigma\,\cup\,\Phi(\mathcal{R})}$.

Proof. For brevity, let $\mathcal{R}'$ denote $\mathcal{R}\mathrm{eq} \cup \mathcal{R}$. The assertion is proven by induction on the length of an $\mathcal{R}'$-proof. Evidently, if $\langle t \approx t', \Sigma(t \approx t')\rangle$ is a member of an $\mathcal{R}'$-proof then clearly $\Sigma \vdash^{\mathcal{R}_{\mathrm{AG}}} \langle t \approx t', (\Sigma \cup \Phi(\mathcal{R}))(t \approx t')\rangle$. Let $\langle t \approx t', a\rangle$ be a member of an $\mathcal{R}'$-proof resulting from previous members by some $R \in \mathcal{R}'$. We proceed for each $R \in \mathcal{R}'$ separately:

(ERef): Apply (ARef) of $\mathcal{R}_{\mathrm{AG}}$.

(ESym): Let $\langle t' \approx t, a\rangle$ result from $\langle t \approx t', a\rangle$ by (ESym). By the induction hypothesis, $\Sigma \vdash^{\mathcal{R}_{\mathrm{AG}}} \langle t \approx t', a'\rangle$, where $a \leq a'$. Hence,

	$\delta_1, \ldots, \delta_l,$	
1:	$\langle t \approx t', a'\rangle,$	proof of $\langle t \approx t', a'\rangle$
2:	$\langle\langle t \approx t', a'\rangle \Rightarrow (t' \approx t), a'\rangle,$	axiom (ASym)
3:	$\langle t' \approx t, a'\rangle,$	by (Cut) on $1, 2$

is an $\mathcal{R}_{\mathrm{AG}}$-proof, where $a \leq a'$.

(ETra): Let $\langle t \approx t'', a \otimes b\rangle$ result from $\langle t \approx t', a\rangle$ and $\langle t' \approx t'', b\rangle$ by (ETra). Assuming $\Sigma \vdash^{\mathcal{R}_{\mathrm{AG}}} \langle t \approx t', a'\rangle$ and $\Sigma \vdash^{\mathcal{R}_{\mathrm{AG}}} \langle t' \approx t'', b'\rangle$ with $a \leq a'$ and $b \leq b'$, we have that

	$\delta_1, \ldots, \delta_l,$	
1:	$\langle t \approx t', a'\rangle,$	proof of $\langle t \approx t', a'\rangle$
2:	$\langle\langle t \approx t', a'\rangle \curlywedge \langle t' \approx t'', b'\rangle \Rightarrow (t \approx t''), a' \otimes b'\rangle,$	axiom (ATra)
3:	$\langle\langle t' \approx t'', b'\rangle \Rightarrow (t \approx t''), a' \otimes b'\rangle,$	by (Cut) on $1, 2$
	$\delta'_1, \ldots, \delta'_{l'},$	
4:	$\langle t' \approx t'', b'\rangle,$	proof of $\langle t' \approx t'', b'\rangle$
5:	$\langle t \approx t'', a' \otimes b'\rangle,$	by (Cut) on $3, 4$

is an $\mathcal{R}_{\mathrm{AG}}$-proof of $\langle t \approx t'', a' \otimes b'\rangle$ such that $a \otimes b \leq a' \otimes b'$.

(ERep) is analogous to (ESym).

(ESub) follows by using (Sub) of $\mathcal{R}_{\mathrm{AG}}$ (obviously, $\emptyset(x/r) = \emptyset$).

(ESup): Apply (Sup) of $\mathcal{R}_{\mathrm{AG}}$ for $P = \emptyset$.

Finally, let $\langle t \approx t', a\rangle$ result from $\langle t_1 \approx t'_1, a_1\rangle, \ldots, \langle t_n \approx t'_n, a_n\rangle$ by some n-ary rule $R \in \mathcal{R}$. Thus, by the induction hypothesis, we can assume that $\Sigma \vdash^{\mathcal{R}_{\mathrm{AG}}} \langle t_i \approx t'_i, a'_i\rangle$, $a_i \leq a'_i$ $(i = 1, \ldots, n)$. Moreover,

$$\big(\Phi(\mathcal{R})\big)(\langle t_1 \approx t'_1, a_1\rangle \curlywedge \cdots \curlywedge \langle t_n \approx t'_n, a_n\rangle \Rightarrow (t \approx t')) = a' \geq a$$

on account of (4). Concatenating the $\mathcal{R}_{\rm AG}$-proofs of all $\langle t_i \approx t'_i, a'_i\rangle$ $(i = 1,\dots,n)$, we can extend the resulting sequence to obtain a proof

$$\begin{array}{rll}
 & \delta_{1,1},\dots,\delta_{1,l_1}, & \\
1: & \langle t_1 \approx t'_1, a'_1\rangle, & \text{proof of } \langle t_1 \approx t'_1, a'_1\rangle \\
 & \vdots \quad \vdots & \vdots \\
 & \delta_{n,1},\dots,\delta_{n,l_n}, & \\
n: & \langle t_n \approx t'_n, a'_n\rangle, & \text{proof of } \langle t_n \approx t'_n, a'_n\rangle \\
n{+}1: & \langle\langle t_1 \approx t'_1, a_1\rangle \mathbin{\&} \cdots \mathbin{\&} \langle t_n \approx t'_n, a_n\rangle \Rightarrow (t \approx t'), a'\rangle, & \text{follows from } \Phi(\mathcal{R}) \\
n{+}2: & \langle\langle t_2 \approx t'_2, a_2\rangle \mathbin{\&} \cdots \mathbin{\&} \langle t_n \approx t'_n, a_n\rangle \Rightarrow (t \approx t'), a'\rangle, & \text{by (Cut) on } 1, n{+}1 \\
 & \vdots \quad \vdots & \vdots \\
2n{+}1: & \langle t \approx t', a'\rangle. & \text{by (Cut) on } n, 2n
\end{array}$$

Hence, $\Sigma \vdash^{\mathcal{R}_{\rm AG}} \langle t \approx t', a'\rangle$ with $a \le a'$. □

Now we turn our attention to the "$\ge$"-part of (7). Observe that an $\mathcal{R}_{\rm AG}$-proof of $\langle t \approx t', a\rangle$ from $\Sigma \cup \Phi(\mathcal{R})$ can contain weighted Horn clauses $\langle P \Rightarrow (s \approx s'), b\rangle$ such that $P \ne \emptyset$. This seems to be a technical complication because we are interested in proving weighted identities, i.e. weighted Horn clauses with $P = \emptyset$. A question that offers itself is whether there is an intuitive explanation of the meaning of the weighted Horn clauses with $P \ne \emptyset$ which can be inferred from $\Sigma \cup \Phi(\mathcal{R})$. We might say, loosely speaking, that for an $\mathbf{L}^*$-deductive system $\mathcal{R}$ of proper $\mathbf{L}^*$-deduction rules, weighted Horn clauses inferred from $\Sigma \cup \Phi(\mathcal{R})$ represent *derived equational deduction rules* which do not strengthen the original $\mathbf{L}^*$-deductive system $\mathcal{R}$. Now we introduce this essential property of weighted Horn clauses more formally.

Definition 4 (composed substitution)**.** A sequence $(x_1/r_1)(x_2/r_2)\cdots(x_k/r_k)$ of substitutions, denoted by τ, is called a *composed substitution.* $t\tau$ abbreviates $t(x_1/r_1)\cdots(x_k/r_k)$, i.e. $t\tau$ is a term resulting from t by the sequential application of substitutions $(x_1/r_1),\dots,(x_k/r_k)$.

Definition 5 ($\boldsymbol{\mathcal{R}}$-admissibility)**.** Let $\mathcal{R}$ denote an equational $\mathbf{L}^*$-deductive system and let Σ be an $\mathbf{L}$-set of identities. Let $\langle P \Rightarrow (t \approx t'), a\rangle$ be a weighted Horn clause with $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle,\dots,\langle t_n, t'_n\rangle\}$. $\langle P \Rightarrow (t \approx t'), a\rangle$ is called *$\mathcal{R}$-admissible for* Σ if for any composed substitution τ and for any truth degrees $a_1,\dots,a_n \in L$ such that $a_1 \ge P(t_1,t'_1),\dots,a_n \ge P(t_n,t'_n)$ we have:

$$\begin{array}{l}
\text{IF } \Sigma \vdash^{\mathcal{R}} \langle t_1\tau \approx t'_1\tau, a_1\rangle \text{ and } \cdots \text{ and } \Sigma \vdash^{\mathcal{R}} \langle t_n\tau \approx t'_n\tau, a_n\rangle,\\
\text{THEN there is } a' \in L \text{ such that } a' \ge a \text{ and } \Sigma \vdash^{\mathcal{R}} \langle t\tau \approx t'\tau, a'\rangle\,.
\end{array}$$

$\mathbf{L}^*$-deduction rule R *preserves the $\mathcal{R}$-admissibility for* Σ if for any weighted Horn clauses $\langle\varphi_1, a_1\rangle,\dots,\langle\varphi_n, a_n\rangle$ which are $\mathcal{R}$-admissible for Σ we have that if $R(\langle\varphi_1, a_1\rangle,\dots,\langle\varphi_n, a_n\rangle) = \langle\varphi, a\rangle$ then $\langle\varphi, a\rangle$ is $\mathcal{R}$-admissible for Σ.

Lemma 6. *Let $\mathcal{R}$ be an equational $\mathbf{L}^*$-deductive system.*

(i) *If $\langle\emptyset \Rightarrow (t \approx t'), a\rangle$ is $\mathcal{R}$-admissible for Σ then $\Sigma \vdash^{\mathcal{R}} \langle t \approx t', a'\rangle$ for some $a' \ge a$.*

(ii) *If $\mathcal{R}$ contains* (ESub) *and if $\Sigma \vdash^{\mathcal{R}} \langle t \approx t', a'\rangle$ then for any $a \le a'$, $\langle\emptyset \Rightarrow (t \approx t'), a\rangle$ is $\mathcal{R}$-admissible for Σ.*

(iii) *If $\mathcal{R}$ contains all rules of $\mathcal{R}$eq then all instances of* (ARef)–(ARep) *are $\mathcal{R}$-admissible for any Σ.*

Proof. (i): Follows directly by definition.

(ii): Let $\Sigma \vdash^{\mathcal{R}} \langle t \approx t', a'\rangle$ and take $a \in L$ such that $a \le a'$. For any composed substitution τ, we can use (ESub) repeatedly to obtain $\Sigma \vdash^{\mathcal{R}} \langle t\tau \approx t'\tau, a'\rangle$. Thus, $\langle\emptyset \Rightarrow (t \approx t'), a\rangle$ is $\mathcal{R}$-admissible for Σ.

(iii): For instance, consider $\langle\langle t \approx t', a\rangle \mathbin{\&} \langle t' \approx t'', b\rangle \Rightarrow (t \approx t''), a \otimes b\rangle$. Let $\Sigma \vdash^{\mathcal{R}} \langle t\tau \approx t'\tau, a'\rangle$ and $\Sigma \vdash^{\mathcal{R}} \langle t'\tau \approx t''\tau, b'\rangle$, where $a' \ge a$ and $b' \ge b$. Then by (ETra) we have $\Sigma \vdash^{\mathcal{R}} \langle t\tau \approx t''\tau, a' \otimes b'\rangle$, where $a' \otimes b' \ge a \otimes b$. Thus, $\langle\langle t \approx t', a\rangle \mathbin{\&} \langle t' \approx t'', b\rangle \Rightarrow (t \approx t''), a \otimes b\rangle$ being an instance of (ATra) is $\mathcal{R}$-admissible for any Σ. One can proceed in a similar way for the other axioms. □

Lemma 7. *For any equational $\mathbf{L}^*$-deductive system $\mathcal{R}$ and any $\mathbf{L}$-set Σ of identities,* (Cut), (Sub), *and* (Wea) *preserve the $\mathcal{R}$-admissibility for Σ.*

Proof. Take an equational $\mathbf{L}^*$-deductive system $\mathcal{R}$ and an $\mathbf{L}$-set Σ of identities. We check that if $\langle\varphi, a\rangle$ was derived using one of the rules (Cut), (Sub), and (Wea) from weighted Horn clauses being $\mathcal{R}$-admissible for Σ, then $\langle\varphi, a\rangle$ is $\mathcal{R}$-admissible for Σ as well.

(Cut): Let $\langle Q \cup P \Rightarrow (t \approx t'), b\rangle$ result from $\langle Q \Rightarrow (s \approx s'), a\rangle$ and $\langle \{c/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle$ by (Cut). Suppose $\langle Q \Rightarrow (s \approx s'), a\rangle$ and $\langle \{c/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle$ are $\mathcal{R}$-admissible for Σ. We show that the resulting weighted Horn clause $\langle Q \cup P \Rightarrow (t \approx t'), b\rangle$ is $\mathcal{R}$-admissible for Σ. Without loss of generality, we shall check only the nontrivial case, i.e. we assume $c \leq a$ and $P(s, s') \leq a$. Let $\mathrm{Supp}(Q \cup P) = \{\langle t_1, t'_1\rangle \ldots, \langle t_n, t'_n\rangle\}$ and let $\Sigma \vdash^{\mathcal{R}} \langle t_i\tau \approx t'_i\tau, a'_i\rangle$, $a'_i \geq (Q \cup P)(t_i, t'_i)$ for each $i = 1, \ldots, n$. Since $Q \subseteq Q \cup P$ and $\langle Q \Rightarrow (s \approx s'), a\rangle$ is $\mathcal{R}$-admissible for Σ, we immediately obtain $\Sigma \vdash^{\mathcal{R}} \langle s\tau \approx s'\tau, a'\rangle$, where $a' \geq a$. Moreover, we have $P \subseteq Q \cup P$ and $a' \geq a \geq c \vee P(s, s')$. Since $\langle \{c/\langle s, s'\rangle\} \cup P \Rightarrow (t \approx t'), b\rangle$ is $\mathcal{R}$-admissible for Σ, we have $\Sigma \vdash^{\mathcal{R}} \langle t\tau \approx t'\tau, b'\rangle$, where $b' \geq b$. Thus, $\langle Q \cup P \Rightarrow (t \approx t'), b\rangle$ is $\mathcal{R}$-admissible for Σ.

(Sub): Take $\langle P \Rightarrow (t \approx t'), a\rangle$ which is $\mathcal{R}$-admissible for Σ and let $\langle P(x/r) \Rightarrow (t(x/r) \approx t'(x/r)), a\rangle$ result from $\langle P \Rightarrow (t \approx t'), a\rangle$ by (Sub). Note that assuming $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle \ldots, \langle t_n, t'_n\rangle\}$, we have

$$\mathrm{Supp}(P(x/r)) = \{\langle t_1(x/r), t'_1(x/r)\rangle \ldots, \langle t_n(x/r), t'_n(x/r)\rangle\}\,.$$

Let $\Sigma \vdash^{\mathcal{R}} \langle t_i(x/r)\tau \approx t'_i(x/r)\tau, a'_i\rangle$ and $\big(P(x/r)\big)(t_i(x/r), t'_i(x/r)) \leq a'_i$ $(i = 1, \ldots, n)$. It follows that

$$P(t_i, t'_i) \leq \bigvee\nolimits_{\substack{s(x/r)=t_i(x/r)\\ s'(x/r)=t'_i(x/r)}} P(s, s') = \big(P(x/r)\big)(t_i(x/r), t'_i(x/r)) \leq a'_i$$

for $i = 1, \ldots, n$. Obviously, $(x/r)\tau$ is a composed substitution. Thus, $\Sigma \vdash^{\mathcal{R}} \langle t(x/r)\tau \approx t'(x/r)\tau, a'\rangle$, where $a' \geq a$. As a consequence, $\langle P(x/r) \Rightarrow (t(x/r) \approx t'(x/r)), a\rangle$ is $\mathcal{R}$-admissible for Σ.

(Wea): Let $\langle P \Rightarrow (t \approx t'), a\rangle$ be $\mathcal{R}$-admissible for Σ and let $\mathrm{Supp}(Q \cup P) = \{\langle t_1, t'_1\rangle \ldots, \langle t_n, t'_n\rangle\}$. Suppose $\langle Q \cup P \Rightarrow (t \approx t'), a\rangle$ results from $\langle P \Rightarrow (t \approx t'), a\rangle$ by (Wea). Let $\Sigma \vdash^{\mathcal{R}} \langle t_i\tau \approx t'_i\tau, a'_i\rangle$ and $(Q \cup P)(t_i, t'_i) \leq a'_i$ $(i = 1, \ldots, n)$. Clearly, $P(t_i, t'_i) \leq (Q \cup P)(t_i, t'_i) \leq a'_i$ for each $\langle t_i, t'_i\rangle \in \mathrm{Supp}(P)$. Hence, $\Sigma \vdash^{\mathcal{R}} \langle t\tau \approx t'\tau, a'\rangle$ for $a' \geq a$ since $\langle P \Rightarrow (t \approx t'), a\rangle$ is $\mathcal{R}$-admissible for Σ. Therefore, $\langle Q \cup P \Rightarrow (t \approx t'), a\rangle$ is $\mathcal{R}$-admissible for Σ. □

Finally, we can prove the "$\geq$"-part of (7).

Lemma 8. $|t \approx t'|^{\mathcal{R}\mathrm{eq} \cup \mathcal{R}}_{\Sigma} \geq |t \approx t'|^{\mathcal{R}_{\mathrm{AG}}}_{\Sigma \cup \Phi(\mathcal{R})}$.

Proof. Let $\mathcal{R}\mathrm{eq} \cup \mathcal{R}$ be abbreviated by $\mathcal{R}'$. First, we check that each $\langle P \Rightarrow (t \approx t'), a\rangle$ being a member of an $\mathcal{R}_{\mathrm{AG}}$-proof from $\Sigma \cup \Phi(\mathcal{R})$ is $\mathcal{R}'$-admissible for Σ. Obviously, if $\langle P \Rightarrow (t \approx t'), a\rangle$ is an instance of (ARef)–(ARep) then it is $\mathcal{R}'$-admissible for Σ, see Lemma 6 (iii).

Let $\langle P \Rightarrow (t \approx t'), a\rangle$ be a member of an $\mathcal{R}_{\mathrm{AG}}$-proof, where $a = \big(\Sigma \cup \Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t'))$. Suppose we have $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle \ldots, \langle t_n, t'_n\rangle\}$ and let $\Sigma \vdash^{\mathcal{R}'} \langle t_i\tau \approx t'_i\tau, a'_i\rangle$, where $P(t_i, t'_i) \leq a'_i$ $(i = 1, \ldots, n)$. We are going to prove (i) $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, b\rangle$ where $b \geq \big(\Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t'))$; and (ii) $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, c\rangle$ where $c \geq \Sigma(P \Rightarrow (t \approx t'))$.

Ad (i): The claim is trivial for $\big(\Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t')) = 0$, so let $\big(\Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t')) \neq 0$. Since $\mathbf{L}$ is finite, there are rules $R_1, \ldots, R_k \in \mathcal{R}$ and permutations $\pi_1, \ldots, \pi_k \in \Pi_n$ such that $R_j^{\pi_j}(P) = \langle t \approx t', b_j\rangle$ $(j = 1, \ldots, k)$, and $\big(\Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t')) = b_1 \vee \cdots \vee b_k$. Since all R_j's are proper, we can conclude that

$$R_j\big(\langle t_{\pi_j(1)}\tau \approx t'_{\pi_j(1)}\tau, a'_{\pi_j(1)}\rangle, \ldots, \langle t_{\pi_j(n)}\tau \approx t'_{\pi_j(n)}\tau, a'_{\pi_j(n)}\rangle\big) = \big\langle t\tau \approx t'\tau, b'_j\big\rangle\,,$$

where $b'_j \geq b_j$ $(j = 1, \ldots, k)$. Therefore, $\Sigma \vdash^{\mathcal{R}'} \big\langle t\tau \approx t'\tau, b'_j\big\rangle$ for each $j = 1, \ldots, k$. Moreover, (ESup) of $\mathcal{R}\mathrm{eq}$ yields $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, b\rangle$, where $b = b'_1 \vee \cdots \vee b'_k \geq b_1 \vee \cdots \vee b_k = \big(\Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t'))$.

Ad (ii): If $P \neq \emptyset$ then $\Sigma(P \Rightarrow (t \approx t')) = 0$ and the claim is trivial. Otherwise, put $c = \Sigma(P \Rightarrow (t \approx t'))$, i.e. $c = \Sigma(t \approx t')$. Clearly, (ESub) gives $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, c\rangle$.

Putting claims (i) and (ii) together, (ESup) yields $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, b \vee c\rangle$, where

$$b \vee c \geq \big(\Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t')) \vee \Sigma(P \Rightarrow (t \approx t')) = \big(\Sigma \cup \Phi(\mathcal{R})\big)(P \Rightarrow (t \approx t')) = a,$$

showing that $\langle P \Rightarrow (t \approx t'), a\rangle$ is $\mathcal{R}'$-admissible for Σ.

From Lemma 7 it follows that (Cut), (Sub), and (Wea) preserve the $\mathcal{R}'$-admissibility for Σ. Thus, it remains to show that (Sup) preserves the $\mathcal{R}'$-admissibility for Σ as well. Consider $P \Rightarrow (t \approx t')$, where $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$. Let $\langle P \Rightarrow (t \approx t'), a\rangle$ and $\langle P \Rightarrow (t \approx t'), b\rangle$ be $\mathcal{R}'$-admissible for Σ. Suppose $\Sigma \vdash^{\mathcal{R}'} \langle t_i\tau \approx t'_i\tau, a'_i\rangle$ with $a'_i \geq P(t_i, t'_i)$ $(i = 1, \ldots, n)$. Then clearly, $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, a'\rangle$ and $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, b'\rangle$ for some $a', b' \in L$ satisfying $a' \geq a$ and $b' \geq b$. Since $\mathcal{R}'$ contains (ESup), it follows that $\Sigma \vdash^{\mathcal{R}'} \langle t\tau \approx t'\tau, a' \vee b'\rangle$, where $a' \vee b' \geq a \vee b$. Hence, $\langle P \Rightarrow (t \approx t'), a \vee b\rangle$ is $\mathcal{R}'$-admissible for Σ.

Altogether, we have shown that if $\Sigma \cup \Phi(\mathcal{R}) \vdash^{\mathcal{R}_{\mathrm{AG}}} \langle P \Rightarrow (t \approx t'), a\rangle$ then $\langle P \Rightarrow (t \approx t'), a\rangle$ is $\mathcal{R}'$-admissible for Σ. Therefore, putting $P = \emptyset$, Lemma 6 (i) yields that $\Sigma \cup \Phi(\mathcal{R}) \vdash^{\mathcal{R}_{\mathrm{AG}}} \langle \emptyset \Rightarrow (t \approx t'), a\rangle$ implies $\Sigma \vdash^{\mathcal{R}'} \langle t \approx t', a'\rangle$ for some $a' \geq a$, which proves the claim of Lemma 8. □

The completeness theorem presented in Section 3 is now a consequence of Lemma 3, Lemma 8, and the completeness of fuzzy Horn logic, see [6].

Remark. (1) In the original paper on FHL [6], we used a general notion of a *proper family of premises* to formalize various restrictions on premises and we showed that for certain families one can have a complete fuzzy Horn logic for wider classes of structures of truth degrees. For instance, this is the case of so-called *crisp premises* (each identity either belongs to a set of premises in degree 1 or in degree 0). On the contrary, the present approach to extended FEL uses Horn clauses with *unrestricted premises* (i.e., an identity belongs to premises in an arbitrary degree taken from L). Such formulas can be used to represent proper equational $\mathbf{L}^*$-deduction rules. From this point of view, the present paper shows the importance of Horn clauses with unrestricted truth-weighted premises. An open question is what equational $\mathbf{L}^*$-deduction rules (if any) are representable by weighted Horn clauses with restricted premises (e.g., the crisp ones).

(2) Let us comment on the role of the chosen truth stresser. In fuzzy Horn logic, each truth stresser influences both the syntactic and semantic entailment. Namely, on the syntax level it is used as a thresholding function for deduction rules while on the semantics level it determines the interpretation of formulas. The choice of a truth stresser does not influence the syntactic and semantic entailment provided that we restrict ourselves only to identities. In the present paper we took the advantage of globalization because of its desirable relationship to the interpretation of $\mathbf{L}^*$-deduction rules. So, another open problem is if one can develop extended FEL using other truth stressers. Note that the role of globalization seems to be essential from the viewpoint of other problems considered earlier, see [7].

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