A NOTE ON RESIDUATED LATTICES WITH GLOBALIZATION

Ivan Chajda[1], Vilém Vychodil[2]

[1]Dept. Algebra and Geometry, Palacký University, Tomkova 40,
CZ-779 00 Olomouc, CZECH REPUBLIC
e-mail: chajda@inf.upol.cz

[2]Dept. Computer Science, Palacký University, Tomkova 40,
CZ-779 00 Olomouc, CZECH REPUBLIC
e-mail: vilem.vychodil@upol.cz

Abstract: It is shown that each residuated lattice equipped with globalization is a ternary discriminator algebra.
AMS Subj. Classification: 03B52, 08A40
Key Words: globalization, fuzzy logic, residuated lattice

Residuated lattices, being initially developed in the 1930s by Dilworth [3, 9] in ring theory, were introduced into the context of fuzzy logic by Goguen [4, 5]. Fundamental contribution to formal fuzzy logic using residuated lattices as the structures of truth degrees is due to Pavelka [7] (logic with evaluated syntax) and Hájek [6] (logic with syntax in the classical style). Currently, residuated lattices play a crucial role in fuzzy logic in both the narrow and the wide sense. However, the common fuzzy logical calculi, such as Hájek's BL-logic [6] (propositional and predicate), are often too weak from the point of view of applications. In addition to so-called schematic extensions of BL-logic allowing us to extend BL-logic by additional axiom schemes, Hájek [6] proposed an extension of (propositional/predicate) BL-logic which results by adding a new unary logical connective "fully true" (this extension turned out to be one of the most important ones). The present paper shows that each residuated lattice endowed with globalization (Takeuti and Titani [8]), which can be seen as a particular truth function of the logical connective "fully true", is a ternary discriminator algebra. Therefore, from the semantic point of view, each finite residuated lattice with globalization is functionally complete. This includes, among other structures, finite linearly ordered BL-algebras extended by the so-called Baaz's operation [1, 6], widely used in applications.

Residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice,
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid,
(iii) $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$
(so-called *adjointness property*).

Having a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ given, by *globalization on* L we mean a unary operation $\triangle$ on L such that

Received: Month, Date, Year

$$\triangle(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By a *residuated lattice with globalization* will be meant a residuated lattice $\mathbf{L}$ endowed with globalization defined on L, i.e. $\triangle$ is the fundamental operation of $\mathbf{L}$.

The following facts follow immediately from the adjointness property (see Bělohlávek [2] and Hájek [6]):

(F1) $a \le b$ iff $a \to b = 1$ $(a, b \in L)$,
(F2) $1 \to a = a$ $(a \in L)$.

Definition 1 Consider two terms

$$g(x,y) = \triangle((x \to y) \wedge (y \to x)),$$
$$h(x,y) = g(x,y) \to 0.$$

For each residuated lattice with globalization $\mathbf{L}$ let $g^{\mathbf{L}}$ and $h^{\mathbf{L}}$ denote the binary term functions on L induced by $g(x,y)$ and $h(x,y)$, respectively.

Lemma 1 *Let $g^{\mathbf{L}}$ and $h^{\mathbf{L}}$ be the above defined functions. Then*

$$g^{\mathbf{L}}(a,a) = 1, \quad h^{\mathbf{L}}(a,a) = 0, \quad \text{and}$$
$$g^{\mathbf{L}}(a,b) = 0, \quad h^{\mathbf{L}}(a,b) = 1, \quad \text{for } a \ne b.$$

Proof. By (F1), $a \le b$ iff $a \to b = 1$, and $b \le a$ iff $b \to a = 1$. Hence, $a = b$ yields $g^{\mathbf{L}}(a,a) = \triangle((a \to a) \wedge (a \to a)) = 1$. If $a \ne b$ then $a \not\le b$ or $b \not\le a$ thus $a \to b \ne 1$ or $b \to a \ne 1$. Therefore, $g^{\mathbf{L}}(a,b) = \triangle((a \to b) \wedge (b \to a)) = 0$ for $a \ne b$. Furthermore, $h^{\mathbf{L}}(a,a) = g^{\mathbf{L}}(a,a) \to 0 = 1 \to 0 = 0$ by (F2). For $a \ne b$ we have $g^{\mathbf{L}}(a,b) = 0$ whence $h^{\mathbf{L}}(a,b) = g^{\mathbf{L}}(a,b) \to 0 = 0 \to 0 = 1$ by (F1).

By the *discriminator function* on $\mathbf{L}$ is meant a function $t^{\mathbf{L}} \colon L^3 \mapsto L$ defined by

$$t(a,b,c) = \begin{cases} a & \text{if } a \ne b, \\ c & \text{if } a = b. \end{cases}$$

A ternary term $t(x,y,z)$ representing the discriminator function on $\mathbf{L}$ is called a *discriminator term* for $\mathbf{L}$, see Werner [10].

Theorem 1 *Each residuated lattice with globalization $\mathbf{L}$ is a ternary discriminator algebra where the discriminator term is $t(x,y,z) = (g(x,y) \to z) \wedge (h(x,y) \to x)$.*

Proof. For the term function $t^{\mathbf{L}}$ on L induced by $t(x,y,z)$ we have $t^{\mathbf{L}}(a,a,c) = (g^{\mathbf{L}}(a,a) \to c) \wedge (h^{\mathbf{L}}(a,a) \to a) = (1 \to c) \wedge (0 \to a) = c \wedge 1 = c$. If $a \ne b$ then $t^{\mathbf{L}}(a,b,c) = (g^{\mathbf{L}}(a,b) \to c) \wedge (h^{\mathbf{L}}(a,b) \to a) = (0 \to c) \wedge (1 \to a) = 1 \wedge a = a$.

Corollary 1 *Let $\mathbf{L}$ be a finite residuated lattice with globalization. Then $\mathbf{L}$ is functionally complete, i.e. any n-ary function $L^n \mapsto L$ is a polynomial function of $\mathbf{L}$.*

Corollary 2 *Let $\mathbf{L}$ be a finite residuated lattice with globalization. Then $\otimes$ is a polynomial function in $\to$, $\wedge$, $\triangle$, and 0.*

Theorem 2 *Let* $\mathbf{L}$ *be a residuated lattice with globalization. Then*

$$a \otimes b = \bigwedge\nolimits_{c \in L} \big(\triangle(a \to (b \to c)) \to c\big).$$

Moreover, if $\mathbf{L}$ *is finite with* $L = \{c_1, \ldots, c_n\}$ *then* $\otimes$ *is represented by the polynomial*

$$(\triangle(x \to (y \to c_1)) \to c_1) \wedge \cdots \wedge (\triangle(x \to (y \to c_n)) \to c_n).$$

Proof. We have $a \otimes b = \bigwedge\{c \,|\, a \leq b \to c\}$ for each $a, b \in L$, because $a \otimes b$ is the least element of $\{c \,|\, a \leq b \to c\}$ (this is easy to see). Furthermore, one can see that $\triangle(a \to (b \to c)) \to c = c$ if $\triangle(a \to (b \to c)) = 1$ which holds iff $a \leq (b \to c)$. On the other hand, $\triangle(a \to (b \to c)) \to c = 1$ if $a \not\leq (b \to c)$ since then $\triangle(a \to (b \to c)) = 0$. Hence, $a \otimes b = \bigwedge_{c \in L}(\triangle(a \to (b \to c)) \to c)$. The rest is obvious.

Note that the discriminator term $t(x, y, z)$ can be introduced in many ways. For instance, one can easily show that $t(x, y, z) = (g'(x, y) \to z) \otimes (h'(x, y) \to x)$ with $g'(x, y) = \triangle((x \to y) \otimes (y \to x))$ and $h'(x, y) = g'(x, y) \to 0$ is also a discriminator term (hint: use the fact that $a \otimes b \leq a \wedge b$ for all $a, b \in L$). Hence, we have

Corollary 3 *Let* $\mathbf{L}$ *be a finite residuated lattice with globalization. Then* $\wedge$ *and* $\vee$ *are polynomial functions in* $\to$, $\otimes$, $\triangle$, *and* 0.

Acknowledgement

I. Chajda is supported by the Council of Czech Government, J14/98:153100011, V. Vychodil is supported by grant no. B1137301 of the Grant Agency of the Academy of Sciences of Czech Republic.

References

[1] Baaz M.: Infinite-valued Gödel logics with 0-1 projections and relativizations. *GÖDEL '96 – Log. Foundations of Math., Comp. Sci. and Phys.*, LNL vol. 6, Springer-Verlag 1996, 23–33.

[2] Bělohlávek R.: *Fuzzy Relational Systems: Foundations and Principles.* Kluwer Academic/Plenum Publishers, New York, 2002.

[3] Dilworth R. P.: Abstract residuation over lattices, *Bull. Amer. Math. Soc.* **44** (1938), 262–268.

[4] Goguen J. A.: L-fuzzy sets. *J. Math. Anal. Appl.* **18** (1967), 145–174.

[5] Goguen J. A.: The logic of inexact concepts. *Synthese* **18** (1968–9), 325–373.

[6] Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[7] Pavelka J.: On fuzzy logic I, II, III. *Z. Math. Logik Grundlagen Math.* **25** (1979), 45–52, 119–134, 447–464.

[8] Takeuti G., Titani S.: Globalization of intuitionistic set theory. *Ann. Pure Appl. Logic* **33** (1987), 195–211.

[9] Ward M., Dilworth R. P.: Residuated lattices, *Trans. Amer. Math. Soc.* **45** (1939), 335–354.

[10] Werner H.: Discriminator algebras, *Studien zur Algebra und ihre Anwendungen*, Band 6, Akademie-Verlag, Berlin, 1978.