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Threshold Boolean logic

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The paper introduces threshold Boolean logic which uses multiple valued threshold truth degrees and bivalent (two-valued) notions of semantic and syntactic entailment. We prove completeness, discuss equality and similarity issues, show connections to other logics, and characterize elementary classes of models by closure properties.

Keywords: *Boolean logic; Fuzzy logic; Model theory; Noetherian lattice; Ultraproduct; Threshold logic*

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1. Introduction and problem setting

Various types of reasoning with threshold values have been used in many fields of computer science as well as engineering applications. Common examples include models of neural networks (threshold values influence the emission of a signal), fuzzy control (the actual control is decided by the inference mechanism which might use rules like "if certain input is close to threshold, then ..."), knowledge discovery (the retrieved information is considered "interesting" if its confidence value exceeds a given threshold), etc. The importance of reasoning with thresholds calls for a logical treatment. One of the early contributions to this topic, paper by Orlowska (1974), showed an axiomatization of a type of threshold logic which originated in research on a mathematical model of nerve cells. In the present paper we focus on the role of thresholds in fuzzy logic.

The applications of truth degrees as threshold values are important in fuzzy logic in both the wide and narrow sense. For instance, when considering the usual notion of a fuzzy set, we often take into account so-called a-cuts of fuzzy sets which can be seen as "thresholded views on fuzzy sets". In more detail, a fuzzy set M in universe U is introduced (in the naive sense) as a map assigning to each element $\mathrm{u} \in U$ a truth degree $M(\mathrm{u}) \in L$ (L is a suitable scale of truth degrees usually endowed with a complete lattice order $\leq$), meaning that u belongs to M in degree $M(\mathrm{u})$ (so-called membership degree of u in M). For any truth degree

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$a \in L$ and a fuzzy set M in universe U, the a-cut ${}^a/M$ of M is a classical set in U where $u \in^a /M$ iff $M(u) \geq a$. Each a-cut of M thus represents an abstract view of the original fuzzy set M in which we focus on elements whose membership degree to M exceeds the threshold given by (truth degree) $a \in L$. The notion of an a-cut is of a crucial importance because any fuzzy set can be represented by the collection of all its a-cuts and many structural notions of fuzzy sets and fuzzy relations are defined by means of a-cuts (i.e. basic methods of defuzzification are based on a-cuts), see Bělohlávek (2002a,b).

The present paper is motivated by the role of threshold truth degrees in the abstract fuzzy logic. In Pavelka (1979), the author has introduced an abstract logical system for fuzzy logic (see also Gerla (2001), Hájek (1998), Novák *et al.* (1999)) which was inspired by ideas of Goguen (1967, 1968, 1969). One of the benefits of Pavelka's framework is that even if we work on an abstract level, we still get useful (and general) notions which can be then particularized in many ways (truth-functional, probabilistic, ...), see Gerla (2001). Consider, for instance, the semantic entailment: We let Fml be a set of *formulas*. In the general setting, no additional structure on Fml is required—formulas are just abstract objects. A system $\mathcal{S}$ of fuzzy sets $E \in \mathcal{S}$ of formulas is called a *semantics* (for abstract fuzzy logic). Thus, $\mathcal{S}$ is a collection of fuzzy sets in universe Fml which assign interpretations to formulas: for $E \in \mathcal{S}$, $E(\varphi) = a$ can be read "interpretation of φ in E is a (e.g. φ is a-true in E)". A *theory* T is any fuzzy set of formulas. $M \in \mathcal{S}$ is called a *model* of a theory T if $T(\varphi) \leq M(\varphi)$ for each $\varphi \in$ Fml. Now, a theory T can be seen as giving to each formula φ its threshold value $T(\varphi)$, saying "the interpretation of φ in every model of T should be (at least) $T(\varphi)$". Finally, the *degree* $\|\varphi\|_T$ *to which* φ *semantically follows from theory* T is defined by

$$\|\varphi\|_T = \bigwedge \{M(\varphi) \mid M \text{ is a model of } T\}.$$

Taking into account the above-mentioned threshold interpretation of T, $\|\varphi\|_T$ is the greatest lower bound (of interpretations of φ in all models of T) which exceeds threshold $T(\varphi)$. The role of thresholds is far more interesting if we assume our logic to be truth-functional: L is a set of degrees (interpreted as "degrees of truth") equipped with truth functions of (fuzzy) logical connectives (conjunction, implication, ...); Fml is defined recursively using atomic formulas and symbols for logical connectives; interpretation of compound formulas is defined componentwise using the truth functions of logical connectives–for instance, if "$\rightarrow$" is a symbol for logical connective "implication" and if "$\Rightarrow$" is the truth function of $\rightarrow$ in L then for any compound formula $\varphi \rightarrow \psi$ and any $E \in \mathcal{S}$ we require $E(\varphi \rightarrow \psi) = E(\varphi) \Rightarrow E(\psi)$. Propositional and predicate fuzzy logics and their fragments developed in Pavelka-style are truth-functional in the above sense, see Hájek (1998), Novák *et al.* (1999), Bělohlávek (2002a,b), Bělohlávek and Vychodil (2006a,b,c).

The truth-functionality brings, however, the following epistemic problem. Consider now a theory T, supplied by an expert, which describes some problem domain. If φ is an atomic formula, the intended meaning of $T(\varphi)$ is more or less clear and intuitively acceptable: the expert observes φ to be true at least in the (threshold) degree $T(\varphi)$. However, from the epistemic point of view, the interpretation of $T(\varphi)$ if φ is a compound formula is questionable. For instance, if φ is a complex propositional formula and $L = [0, 1]$ with its natural ordering, what can we say about $T(\varphi) = 0.5$? Apparently, such a question is a nontrivial one: the intuitive interpretation of $T(\varphi) = 0.5$ depends on the chosen structure of truth degrees (truth functions of logical connectives) and on our interpretation of intermediate truth degrees. Even if we fix a particular structure, the question is still hard to answer because fuzzy logical connectives do not have a straightforward interpretation as the classical (two-valued) ones. Everybody is familiar with the truth function of the classical implication: $(0 \Rightarrow 0) = (0 \Rightarrow 1) = (1 \Rightarrow 1) = 1$, and $(1 \Rightarrow 0) = 0$ with the intuitive meaning "if-then". On the contrary, the interpretation of truth functions of fuzzy implications is sort of problematic: for example, the truth function of the standard Łukasiewicz implication satisfies the law of double negation while the truth function of the standard Gödel implication does not. What is then the difference between $T(\varphi \rightarrow \psi) = 0.5$ if we use the Łukasiewicz implication and $T(\varphi \rightarrow \psi) = 0.5$ if we use the Gödel one? What we know for sure is that there should be some difference since the truth functions do not coincide.

In this paper we try to avoid the problem with the intuitive meaning of $T(\varphi)$ as follows. Instead of considering theories as fuzzy sets of formulas, we introduce theories as sets of *threshold formulas*. We keep the idea of assigning a threshold degree to atomic formulas: an atomic formula with a threshold will be denoted $\langle\varphi, a\rangle$ where φ is an atomic formula (in the previous sense) and $a \in L$ is a threshold truth degree, meaning "φ in degree (at least) a". The compound formulas of our logic will be constituted from atomic formulas with thresholds using logical connectives with the classical interpretation. For instance, if $\langle\varphi, a\rangle$ and $\langle\psi, b\rangle$ denote atomic formulas with thresholds and if $\neg$ and $\rightarrow$ are symbols for logical connectives (negation and implication, respectively) then $\neg\langle\varphi, a\rangle$ and $\langle\varphi, a\rangle \rightarrow \langle\psi, b\rangle$ would be compound formulas, saying "*not* φ in degree (at least) a", and "*if* φ in degree (at least) a, *then* ψ in degree (at least) b". It is worth to mention here that the structure of (threshold) truth degrees L becomes now a part of our language because threshold truth degrees are an integral part of our formulas. One of the benefits of our approach is that the intended meaning of compound formulas is clear because we use the classical logical connectives, i.e. $\neg$ and $\rightarrow$ are meant to be interpreted by standard Boolean truth functions of negation and implication.

Following the two-valued nature of our logical connectives, we define the notions of the semantic entailment and syntactic entailment (provability) in much the same way as in the classical logic. For this reason we call our logic threshold *Boolean logic* because the term "fuzzy logic" is usually used for logics with logical connectives interpreted by truth functions defined in particular multiple-valued structures of truth degrees (e.g. residuated lattices) which contain also intermediate truth degrees. Let us stress that, in general, the structure L of threshold truth degrees has nothing to do with the two-valued Boolean algebra which serves us as the implicit structure of truth degrees (in which we interpret logical connectives). Still, our logic is connected with (predicate) fuzzy logics at least at the epistemic level—the models for languages of our logic will be the usual fuzzy structures. Form this point of view, threshold Boolean logic can be seen as a calculus for reasoning about fuzzy structures which uses the classical interpretation of logical connectives.

Our approach is analogous to that of Kifer and Lozinskii (1992). The motivation of Kifer and Lozinskii (1992) was to develop a logic (annotated predicate logic) for reasoning with inconsistency so that the inconsistency of a theory would not be so "destructive" as in the classical case. The overlap of Kifer and Lozinskii (1992) and the present paper is small (for instance, we use definitions similar to that of Kifer and Lozinskii (1992), however, we use different technique to prove completeness and deal with topics which do not have their analogies in Kifer and Lozinskii (1992)), we will comment on this in the paper. Our paper and the paper of Kifer and Lozinskii (1992) sort of complement each other. Another approach which is similar to ours and which uses so-called signed formulas is that of Hähnle (1991), see also Beckert *et al.* (1999).

The paper is organized as follows. In Section 2, we present preliminaries. Section 3 introduces basic notions of the syntax and semantics. Section 4 contains proofs of soundness and completeness. In Section 5, we discuss languages with equality. Section 6 shows a connection to fuzzy Horn logic. In Section 7, we introduce a generalization of the construction of an ultraproduct and prove that model classes can be characterized by closure properties. Section 8 lists open problems.

2. Preliminaries

We are going to use Noetherian lattices as the structures of threshold truth degrees. Noetherian lattices have traditionally been connected with study of the module theory, see Călugăreanu (2000). Recent results of Bělohlávek and Vychodil (2006a,b,c) indicate that Noetherian lattices (possibly endowed with additional operations) might be interesting also from the viewpoint of fuzzy logic. A *Noetherian lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice (i.e. infima and suprema of arbitrary subsets of L exist in L) with 0 and 1 being the least and the greatest elements of L, respectively;
(ii) for any $\{a_i | i \in I\} \subseteq L$ there is finite $J \subseteq I$ such that $\bigvee\{a_i | i \in I\} = \bigvee\{a_j | j \in J\}$.

Recall that each $\mathbf{L}$ induces a partial (lattice) order: we put $a \leq b$ iff $a \wedge b = a$ (or equivalently, $a \vee b = b$). The elements of L will be called *(threshold) truth degrees*. The intuitive meaning of $a \leq b$ is "a is less true than or equally true as b". It is almost evident that a complete lattice $\mathbf{L}$ is Noetherian iff each element of L is compact, see Călugăreanu (2000). Each finite lattice is Noetherian, because (ii) is satisfied trivially. Each finite subset of the real unit interval endowed with the operations of minimum and maximum is a Noetherian lattice. Finite subsets of the real unit interval are perhaps the most commonly used scales of truth degrees in applications. Note that even if most of theoretical methods of fuzzy logic are developed over infinite structures, finite scales of truth degrees are important from the computational point of view. Thus, Noetherian lattices include an important family of structures of truth degrees. The class of Noetherian lattices includes also infinite lattices. For instance, put $L = \{1/n | n \in \mathbb{N}\} \cup \{0\}$ and let $\wedge$ and $\vee$ be minimum and maximum, respectively. Then $\langle L, \wedge, \vee, 0, 1\rangle$ is an infinite (linearly ordered) Noetherian lattice. Unlike otherwise mentioned, we assume that we are given a Noetherian lattice $\mathbf{L}$ which serves as the structure of threshold truth degrees.

Let $U \neq \emptyset$ be a universe set. An n-ary $\mathbf{L}$-*relation* (n-ary *fuzzy relation with truth degrees in* $\mathbf{L}$) R on U is a map $R : U^n \to L$, $R(\mathrm{u}_1, \ldots, \mathrm{u}_n) \in L$ being interpreted as the degree of "elements $\mathrm{u}_1, \ldots, \mathrm{u}_n$ (in this order) are R-related". Unary $\mathbf{L}$-relations on U, i.e. maps $A : U \to L$, will be called $\mathbf{L}$-*sets* in U ($A(\mathrm{u})$ is interpreted as the degree of "element u belongs to A"). If $\mathbf{L}$ is a two-element chain with $0 < 1$, $\mathbf{L}$-sets coincide with sets of the naive set theory (classical sets) because $A : U \to \{0, 1\}$ is the characteristic function of A. For an n-ary $\mathbf{L}$-relation $R : U^n \to L$ and $a \in L$, we define an a-cut ${}^a/R$ of R by putting ${}^a/R = \{\langle \mathrm{u}_1, \ldots, \mathrm{u}_n\rangle \in U^n | a \leq R(\mathrm{u}_1, \ldots, \mathrm{u}_n)\}$. Thus, if R is an n-ary $\mathbf{L}$-relation on U then each a-cut of R is a classical n-ary relation on U. Each $\mathbf{L}$-relation is uniquely determined by the system of all its a-cuts (so-called cutlike representation, see Bělohlávek (2002a,b)).

3. Syntax and semantics of TBL

By a *type* we mean a triplet $\langle R, F, \sigma\rangle$, where R is a set of relation symbols, F is a set of function symbols, $R \cap F = \emptyset$, and σ is a map assigning to each relation and function symbol $s \in F \cup R$ its arity $\sigma(s) \in \mathbb{N}_0$. Given a Noetherian lattice $\mathbf{L}$, and type $\langle R, F, \sigma\rangle$, an **L**-*language of type* $\langle R, F, \sigma\rangle$, denoted by $\mathcal{L}$, is a collection of symbols for logical connectives and quantifiers ($\rightarrow$, $\neg$, and $\forall$), symbols for variables $x, y, \ldots$, relation symbols $r \in R$, function symbols $f \in F$, and auxiliary symbols (parentheses, etc.). In addition to that, we assume that $\mathcal{L}$ also contains all the elements of $\mathbf{L}$ which can be seen as *symbols for threshold truth degrees*. This is where we distinguish ourselves from languages of the classical (predicate) logic and most fuzzy logics as well, because the truth degrees from $\mathbf{L}$ are contained in the **L**-language. In this case, it cannot cause any undesirable mixing of syntax and semantics because we are going to use classical (two-valued) notion of validity. If $\mathbf{L}$ is clear from the context, we call **L**-languages simply *languages*.

Terms (of language $\mathcal{L}$) are defined as usual. That is, each variable of $\mathcal{L}$ is a term of $\mathcal{L}$, and if $t_1, \ldots, t_n$ are terms of $\mathcal{L}$ and f is an n-ary function symbol of $\mathcal{L}$ then $f(t_1, \ldots, t_n)$ is a term of $\mathcal{L}$. *Formulas of* $\mathcal{L}$ are defined as follows:

(i) if $t_1, \ldots, t_n$ are terms of $\mathcal{L}$, r is an n-ary relation symbol of $\mathcal{L}$, and $a \in L$, then $\langle r(t_1, \ldots, t_n), a\rangle$ is an (atomic) formula of $\mathcal{L}$,
(ii) if φ and ψ are formulas of $\mathcal{L}$, then $(\varphi \rightarrow \psi)$ and $\neg\, \varphi$ are formulas of $\mathcal{L}$,
(iii) if x is a variable of $\mathcal{L}$ and φ is a formula of $\mathcal{L}$, then $(\forall x)\varphi$ is a formula of $\mathcal{L}$.

For each atomic formula $\langle r(t_1, \ldots, t_n), a\rangle$, $r(t_1, \ldots, t_n)$ is an atomic formula in the classical sense. Formulas of TBL will be occasionally referred to as *threshold formulas* in order to distinguish formulas of TBL from formulas of the classical predicate logic. TBL uses only $\rightarrow$ and $\neg$ as the basic (symbols for) logical connectives and $\forall$ for the universal quantifier. The other logical connectives (conjunction, disjunction, etc.) and the existential quantifier can be introduced as shorthands for formulas, e.g., $\varphi \;\&\; \psi$ stands for $\neg\, (\varphi \rightarrow \neg\, \psi)$, etc. Furthermore, we assume the usual conventions for writing formulas like the omission of parentheses, see Mendelson (1997). For the sake of brevity, we skip definitions of some notions (e.g. an occurrence of a variable in a formula), because they remain the same as in the classical predicate logic (i.e. the notions are not influenced by threshold truth degrees present in atomic formulas).

The intended meaning of $\langle r(t_1, \ldots, t_n), a\rangle$, which will be fully justified once we introduce the interpretation of formulas, is the following: "t_1 and $\ldots$ and t_n are r-related in degree (at least) a". Given $\langle r(t_1, \ldots, t_n), a\rangle$ and $\langle r(t_1, \ldots, t_n), a'\rangle$ with $a \leq a'$, it is intuitively acceptable to say that if "t_1 and $\ldots$ and t_n are r-related in degree a'" then also "t_1 and $\ldots$ and t_n are r-related in degree a". Loosely speaking, atomic formula $\langle r(t_1, \ldots, t_n), a\rangle$ is weaker than $\langle r(t_1, \ldots, t_n), a'\rangle$ in sense of the intended meaning of threshold truth degrees. This observation will be introduced formally as follows: for formulas φ, ψ of the same language we put $\varphi \trianglelefteq \psi$ if

(a) φ is $\langle r(t_1, \ldots, t_n), a\rangle$, ψ is $\langle r(t_1, \ldots, t_n), b\rangle$, and $a \leq b$; or
(b) φ is $\varphi_1 \rightarrow \varphi_2$, ψ is $\psi_1 \rightarrow \psi_2$, and $\psi_1 \trianglelefteq \varphi_1$ and $\varphi_2 \trianglelefteq \psi_2$; or

(c) φ is $\neg\varphi'$, ψ is $\neg\psi'$, and $\psi' \trianglelefteq \varphi'$; or
(d) φ is $(\forall x)\varphi'$, ψ is $(\forall x)\psi'$, and $\varphi' \trianglelefteq \psi'$.

The relationship $\varphi \trianglelefteq \psi$ can be read as "φ is weaker than ψ". It is easily seen that $\trianglelefteq$ is a partial order on the set of all formulas of a given language. Axiom schemes of TBL consists of the axiom schemes of the classical predicate logic, see Mendelson (1997):

(P1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,
(P2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$,
(P3) $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$,
($\forall$1) $(\forall x)(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow (\forall x)\psi)$ if φ contains no free occurrence of x,
($\forall$2) $(\forall x)\varphi \rightarrow \varphi(x/t)$ if term t is free for x in φ,

and the additional axiom schemes specific to TBL:

(Thr) $\psi \rightarrow \varphi$ if $\varphi \trianglelefteq \psi$,
(Lea) $\langle r(x_1, \ldots, x_n), 0\rangle$,
(Sup) $\langle r(x_1, \ldots, x_n), a\rangle \rightarrow (\langle r(x_1, \ldots, x_n), b\rangle \rightarrow \langle r(x_1, \ldots, x_n), a \vee b\rangle)$

where $a, b \in L$, r is an n-ary relation symbol, and $x_1, \ldots, x_n$ are variables. The deduction rules of TBL are *modus ponens* (from φ and $\varphi \rightarrow \psi$ infer ψ) and *generalization* (from φ infer $(\forall x)\varphi$). A set T of formulas of language $\mathcal{L}$ is called a *theory (of $\mathcal{L}$)*. The notions of a proof and a provability are defined as usual. A sequence $\varphi_1, \ldots, \varphi_n$ is called a *proof* of φ from T if $\varphi = \varphi_n$ and each φ_i $(i = 1, \ldots, n)$ either is an axiom, or belongs to T, or is inferred (in a single step) by modus ponens or generalization from formula(s) which are among $\varphi_1, \ldots, \varphi_{i-1}$. *$\varphi$ is provable from T*, denoted by $T \vdash \varphi$, if there is a proof of φ from T.

The intuitive meaning of axiom schemes (Thr), (Lea) and (Sup) is the following. (Thr) ensures that if ψ is provable from T then each weaker formula $\varphi \trianglelefteq \psi$ is also provable from T. (Lea) says that "each $r(x_1, \ldots, x_n)$ holds at least in degree 0". Scheme (Sup) describes the following property: "if $r(x_1, \ldots, x_n)$ holds in degree at least a and if $r(x_1, \ldots, x_n)$ holds in degree as least b then $r(x_1, \ldots, x_n)$ holds in degree at least $a \vee b$". The meaning of (Thr), (Lea), and (Sup) will be apparent once we introduce the semantics of TBL.

In order to introduce the interpretation of formulas of TBL, we need suitable semantic structures. Having given a language $\mathcal{L}$ of type $\langle R, F, \sigma\rangle$, an **L**-*structure for language $\mathcal{L}$* is a triplet $\mathbf{M} = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ such that $M \neq \emptyset$,

$$R^{\mathbf{M}} = \{r^{\mathbf{M}} : M^n \rightarrow L \mid r \in R \quad \text{and} \quad \sigma(r) = n\}, \quad \text{and}$$

$$F^{\mathbf{M}} = \{f^{\mathbf{M}} : M^n \rightarrow M \mid f \in F \quad \text{and} \quad \sigma(f) = n\}.$$

That is, **M** represents a nonempty universe set M endowed with a collection $R^{\mathbf{M}}$ of **L**-relations and a collection $F^{\mathbf{M}}$ of functions so that each n-ary relation symbol $r \in R$ corresponds with (is interpreted by) an n-ary **L**-relation $r^{\mathbf{M}} \in R^{\mathbf{M}}$ and each n-ary function symbol $f \in F$ corresponds with (is interpreted by) an n-ary function $f^{\mathbf{M}} \in F^{\mathbf{M}}$. If **L** is the two-element Boolean algebra then the **L**-structures are but the classical structures of predicate logic.

An **M**-*valuation* is a mapping v assigning to each variable x an element of M. For **M**-valuations v, w we write $v \equiv_x w$ if $v(y) = w(y)$ for each variable y distinct from x. If **M** is clear from the context, **M**-valuations shall be simply called valuations. Given an **M**-valuation v, the interpretation $\|t\|_{\mathbf{M},v}$ *of a term t in* **M** *under* v is defined as usual, i.e. $\|t\|_{\mathbf{M},v} = v(x)$ if t is a variable x, and $\|t\|_{\mathbf{M},v} = f^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v})$ if t is $f(t_1, \ldots, t_n)$.

φ *is true in an* **L**-*structure* **M** *under an* **M**-*valuation* v, written $M \models \varphi[v]$, if

(i) φ is $\langle r(t_1, \ldots, t_n), a\rangle$, and $r^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v}) \geq a$; or
(ii) φ is $\psi \rightarrow \chi$, and if $\mathbf{M} \models \psi[v]$ then $\mathbf{M} \models \chi[v]$; or
(iii) φ is $\neg\, \psi$, and $\mathbf{M} \not\models \psi[v]$ (i.e. ψ is not true in **M** under v); or
(iv) φ is $(\forall x)\psi$, and $\mathbf{M} \models \psi[w]$ for each **M**-valuation $w \equiv_x v$.

φ *holds (is valid)* in **M** if $\mathbf{M} \models \varphi[v]$ for each **M**-valuation v. Observe that the above definition of $\mathbf{M} \models \varphi[v]$ differs from the classical one only in (i), because TBL uses atomic formulas with threshold degrees. That is, TBL uses logical connectives $\rightarrow$ and $\neg$ with the classical interpretation.

Finally, we need notions of a model and a semantic entailment. Given a theory T of $\mathcal{L}$, an **L**-structure **M** for $\mathcal{L}$ is called a *model of T*, written $\mathbf{M} \in \mathrm{Mod}(T)$, if each $\varphi \in T$ holds in **M**. φ *semantically follows from T*, denoted by $T \models \varphi$, if φ holds in each model of T. In the following section we prove that TBL is complete. That is, we prove that the syntactic entailment $\vdash$ (provability) coincides with the semantic entailment $\models$ (validity in models).

Remarks. Let us now comment on the approach in Kifer and Lozinskii (1992), where the authors presented the so-called *annotated predicate calculus* (APC). Formulas of TBL can be thought of as particular formulas of APC. Indeed, threshold formulas are basically formulas of APC which use threshold truth degrees as annotations. The interpretation of formulas of TBL is defined the same way as that of annotated formulas. From the point of view of Kifer and Lozinskii (1992), TBL uses only the so-called ontological interpretation of logical connectives. However, TBL differs from APC in several aspects. The basic structures of annotations in APC are *belief upper semilattices* whose elements (annotations) are interpreted as “strengths of (observer's) belief”. The basic completeness theorem in Kifer and Lozinskii (1992) is based on the resolution method and it is restricted to finite belief semilattices. Our basic structures of truth values (i.e. annotations in terms of APC) are Noetherian lattices whose elements are interpreted as it is usual in fuzzy logic (truth values are “the degrees of truth”). The present approach uses axiomatization which is close to that of Hilbert-style logics, and the completeness is proven for each Noetherian lattice (this includes infinite structures of truth degrees).

4. Soundness and completeness of TBL

We are going to prove soundness and completeness of TBL. Soundness is more or less clear and routine to check. Completeness will be proven by embedding TBL into the classical predicate logic (PC). In order to show soundness, we need the following lemma.

LEMMA 1. If $\varphi \trianglelefteq \psi$ and $\mathbf{M} \models \psi[v]$ then $\mathbf{M} \models \varphi[v]$.

Proof. We prove the claim by structural induction. Let φ be $\langle r(t_1, \ldots, t_n), a\rangle$, ψ be $\langle r(t_1, \ldots, t_n), b\rangle$, where $a \leq b$. If $\mathbf{M} \models \psi[v]$ then $r^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v}) \geq b \geq a$, i.e. $\mathbf{M} \models \varphi[v]$. Let φ be $\varphi_1 \to \varphi_2$, ψ be $\psi_1 \to \psi_2$, and let $\psi_1 \trianglelefteq \varphi_1$ and $\varphi_2 \trianglelefteq \psi_2$. Suppose $\mathbf{M} \models \psi[v]$, i.e. if $\mathbf{M} \models \psi_1[v]$ then $\mathbf{M} \models \psi_2[v]$. By induction hypothesis, if $\mathbf{M} \models \varphi_1[v]$ then $\mathbf{M} \models \psi_1[v]$; if $\mathbf{M} \models \psi_2[v]$ then $\mathbf{M} \models \varphi_2[v]$. Putting it together, we have that if $\mathbf{M} \models \varphi_1[v]$ then $\mathbf{M} \models \varphi_2[v]$, i.e. $\mathbf{M} \models \varphi[v]$. Let φ be $\neg\, \varphi'$, ψ be $\neg\, \psi'$, and let $\psi' \trianglelefteq \varphi'$. If $\mathbf{M} \models \psi[v]$ then $\mathbf{M} \not\models \psi'[v]$. Thus, $\mathbf{M} \not\models \varphi'[v]$ which gives $\mathbf{M} \models \varphi[v]$. Finally, let φ be $(\forall x)\varphi'$, ψ be $(\forall x)\psi'$, and let $\varphi' \trianglelefteq \psi'$. If $\mathbf{M} \models \psi'[v]$ then $\mathbf{M} \models \psi'[w]$ for each $w \equiv_x v$. By induction hypothesis, $\mathbf{M} \models \varphi'[w]$ for each $w \equiv_x v$, i.e. $\mathbf{M} \models \varphi'[v]$. □

It is apparent that each axiom of the form (P1)–(∀2) is true in each **L**-structure because we use classical logical connectives. In addition to that, (Lea) and (Sup) are true in each **L**-structure because **L** is a lattice with the least element 0. Finally, (Thr) is true in each **L**-structure on account of Lemma 1. Now, by induction on the length of a proof, one can show that each formula φ which is provable from T follows semantically from T. That proves the soundness of TBL.

We now turn our attention to completeness. The key idea of the subsequent proof is that we express provability in TBL by provability in PC and then show that TBL and PC also agree on the semantics level. Once we show that, completeness of TBL will be a consequence of the completeness of PC.

Let us introduce the following convention. For an **L**-language $\mathcal{L}$ of type $\langle R, F, \sigma\rangle$ we consider a language $\mathcal{L}_{\mathbf{L}}$ of type $\langle R_{\mathbf{L}}, F, \sigma_{\mathbf{L}}\rangle$, where $R_{\mathbf{L}} = \{r_a | r \in R \text{ and } a \in L\}$, $\sigma_{\mathbf{L}}(r_a) = \sigma(r)$ for each $r_a \in R_{\mathbf{L}}$, and $\sigma_{\mathbf{L}}(f) = \sigma(f)$ for each $f \in F$. That is, $\mathcal{L}_{\mathbf{L}}$ is a language resulting from $\mathcal{L}$ by replacing each n-ary relation symbol r by a collection of n-ary relation symbols r_a ($a \in L$). Note that each $\mathcal{L}_{\mathbf{L}}$ can be seen as a classical predicate language (if we omit the symbols for threshold truth degrees).

For each threshold formula φ of language $\mathcal{L}$ we consider a classical formula φ° of language $\mathcal{L}_{\mathbf{L}}$ as follows:

$$\varphi^\circ = \begin{cases} r_a(t_1, \ldots, t_n) & \text{if } \varphi \text{ is } \langle r(t_1, \ldots, t_n), a\rangle, \\ (\psi^\circ \to \chi^\circ) & \text{if } \varphi \text{ is } (\psi \to \chi), \\ \neg\, \psi^\circ & \text{if } \varphi \text{ is } \neg\, \psi, \\ (\forall x)\psi^\circ & \text{if } \varphi \text{ is } (\forall x)\psi, \end{cases}$$

i.e. φ° results from φ by replacing all atomic subformulas $\langle r(t_1, \ldots, t_n), a\rangle$ of φ by classical atomic formulas $r_a(t_1, \ldots, t_n)$. For a theory T of threshold formulas of language $\mathcal{L}$ we denote by T° the (classical) theory defined by $T^\circ = \{\varphi^\circ | \varphi \in T\}$. In the sequel, let T°_L denote the theory of language $\mathcal{L}_{\mathbf{L}}$ which consists of all the instances of (Thr), (Lea), and (Sup) converted to classical formulas using "∘", i.e. T°_L consists of formulas $\psi^\circ \to \varphi^\circ$ such that $\varphi \trianglelefteq \psi$, formulas $r_0(x_1, \ldots, x_n)$, and formulas $r_a(x_1, \ldots, x_n) \to (r_b(x_1, \ldots, x_n) \to r_{a \vee b}(x_1, \ldots, x_n))$, where $a, b \in L$, $r \in R$, and $x_1, \ldots, x_n$ are variables. Since $T^\circ \cup T^\circ_L$ is a classical predicate theory, we can consider the classical syntactic and semantic entailment ($\vdash$ and $\models$) from $T^\circ \cup T^\circ_L$, see Chang and Keisler (1973), Mendelson (1997). For brevity, each ordinary predicate structure which is a model of T° and/or T°_L will be called a *classical model* of T° and/or T°_L.

LEMMA 2. Let $\mathcal{L}$ be a language of type $\langle R, F, \sigma\rangle$, $\mathbf{M}$ be a classical model of T_L°. Then for each n-ary $r \in R$ we have

(i) $a \leq b$ implies $r_b^{\mathbf{M}} \subseteq r_a^{\mathbf{M}}$,
(ii) for any $u_1, \ldots, u_n \in M$, $\{a \in L | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$ has a greatest element,
(iii) $b \leq \bigvee \{a \in L | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$ iff $\langle u_1, \ldots, u_n\rangle \in r_b^{\mathbf{M}}$.

Proof

(i) Let $a \leq b$. The classical counterpart of (Thr) holds in $\mathbf{M}$ because $\mathbf{M} \in \mathrm{Mod}(T_L^\circ)$. Therefore, from $\mathbf{M} \models r_b(x_1, \ldots, x_n) \to r_a(x_1, \ldots, x_n)$ it follows that $\langle u_1, \ldots, u_n\rangle \in r_b^{\mathbf{M}}$ implies $\langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}$, i.e. $r_b^{\mathbf{M}} \subseteq r_a^{\mathbf{M}}$.
(ii) First, $0 \in \{a | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$ due to the classical counterpart of (Lea) which holds in $\mathbf{M}$. Put $c = \bigvee \{a | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$. Since $\mathbf{L}$ is a Noetherian lattice and $\{a | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$ is nonempty, there are $a_1, \ldots, a_k \in L$ such that $c = a_1 \vee \ldots \vee a_k$, and $\langle u_1, \ldots, u_n\rangle \in r_{a_i}^{\mathbf{M}}$ for each $i = 1, \ldots, k$. Now the classical counterpart of (Sup) gives $\langle u_1, \ldots, u_n\rangle \in r_{a_1 \vee \cdots \vee a_k}^{\mathbf{M}} = r_c^{\mathbf{M}}$. Therefore, $c \in \{a | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$, i.e. c is the greatest element of $\{a | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$.
(iii) The "$\Leftarrow$"-part is trivial. Conversely, $c = \bigvee \{a | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$ is the greatest element of $\{a | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$ due to (ii). Therefore, $\langle u_1, \ldots, u_n\rangle \in r_c^{\mathbf{M}}$. Then we have $b \leq c$, and thus (i) yields $\langle u_1, \ldots, u_n\rangle \in r_b^{\mathbf{M}}$. □

Let us comment on the consequences of Lemma 2. Suppose $\mathbf{M}$ is a classical model of T_L°. From Lemma 2 it follows that each L-indexed system $\{r_a^{\mathbf{M}} | a \in L\}$ of relations is an $\mathbf{L}$-nested system of relations, see Bělohlávek (2002b). Consequently, $\{r_a^{\mathbf{M}} | a \in L\}$ is the cutlike representation of the $\mathbf{L}$-relation R defined so that $R(u_1, \ldots, u_n)$ is the greatest element of $\{a \in L | \langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}}\}$. This property will be used in the proof of the following theorem.

THEOREM 3 (COMPLETENESS OF TBL). For each theory T of language $\mathcal{L}$ and for any formula φ of that language we have $T \vdash \varphi$ iff $T \models \varphi$.

Proof. The proof is done by showing that

$$T \vdash \varphi \text{ iff } T^\circ \cup T_L^\circ \vdash \varphi^\circ \text{ iff } T^\circ \cup T_L^\circ \models \varphi^\circ \text{ iff } T \models \varphi$$

First, "$T \vdash \varphi$ iff $T^\circ \cup T_L^\circ \vdash \varphi^\circ$" can be easily proven by induction on the length of proof because both PC and TBL use the same deduction rules and the same collection of axioms schemes (recall that the instances of (Thr), (Lea), and (Sup) are contained in T_L°). Moreover, "$T^\circ \cup T_L^\circ \vdash \varphi^\circ$ iff $T^\circ \cup T_L^\circ \models \varphi^\circ$" follows from the completeness of PC. So it remains to show "$T^\circ \cup T_L^\circ \models \varphi^\circ$ iff $T \models \varphi$".

"If $T^\circ \cup T_L^\circ \models \varphi^\circ$ then $T \models \varphi$": First we show that for each $\mathbf{L}$-structure $\mathbf{M}$ for language $\mathcal{L}$ there is a classical structure $\mathbf{M}^\circ$ for language $\mathcal{L}_{\mathrm{L}}$ such that for any threshold formula ψ we have $\mathbf{M} \models \psi$ iff $\mathbf{M}^\circ \models \psi^\circ$. $\mathbf{M}^\circ$ can be found as follows: let $\mathbf{M}^\circ$ be the classical structure whose functional part coincides with the functional part of $\mathbf{M}$, i.e. $\mathbf{M}^\circ$ has the same universe set and functions as $\mathbf{M}$ has; for any n-ary $r \in R$ and a degree $a \in L$ we put $\langle u_1, \ldots, u_n\rangle \in r_a^{\mathbf{M}^\circ}$ iff

$r^{\mathbf{M}}(u_1, \ldots, u_n) \geq a$. By structural induction one can show that $\mathbf{M} \models \psi[v]$ iff $\mathbf{M}^\circ \models \psi^\circ[v]$ for each valuation v (the proof is routine and therefore omitted). Moreover, since each instance of (Thr), (Lea), and (Sup) is valid in $\mathbf{M}$, we get that $\mathbf{M}^\circ \in \mathrm{Mod}(T_L^\circ)$. As a consequence, for each model $\mathbf{M} \in \mathrm{Mod}(T)$ there is a classical model $\mathbf{M}^\circ \in \mathrm{Mod}(T^\circ \cup T_L^\circ)$ such that $\mathbf{M} \models \psi$ iff $\mathbf{M}^\circ \models \psi^\circ$ for any ψ which further gives that if $T^\circ \cup T_L^\circ \models \varphi^\circ$ then $T \models \varphi$.

"If $T \models \varphi$ then $T^\circ \cup T_L^\circ \models \varphi^\circ$": Analogously as above, we show that for each classical model $\mathbf{M} \in \mathrm{Mod}(T_L^\circ)$ there is an $\mathbf{L}$-structure $\mathbf{N}$ for $\mathcal{L}$ such that $\mathbf{N} \models \psi$ iff $\mathbf{M} \models \psi^\circ$ for any ψ. Let $\mathbf{M} \in \mathrm{Mod}(T_L^\circ)$. We consider an $\mathbf{L}$-structure $\mathbf{N}$ for $\mathcal{L}$ whose functional part coincides with that of $\mathbf{M}$ and for any n-ary $r \in R$ we put

$$r^{\mathbf{N}}(u_1, \ldots, u_n) = \bigvee \{a \in L | \langle u_1, \ldots, u_n \rangle \in r_a^{\mathbf{M}}\}.$$

Now take a formula ψ and a valuation v. If ψ is of the form $\langle r(t_1, \ldots, t_n), b\rangle$ we can proceed as follows: we have $\mathbf{N} \models \langle r(t_1, \ldots, t_n), b\rangle[v]$ iff $r^{\mathbf{N}}(\|t_1\|_{\mathbf{N},v}, \ldots, \|t_n\|_{\mathbf{N},v}) \geq b$ iff $r^{\mathbf{N}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v}) \geq b$ iff $\bigvee \{a \in L | \langle \|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v}\rangle \in r_a^{\mathbf{M}}\} \geq b$. Lemma 2 (iii) yields that the last inequality is true iff $\langle \|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v}\rangle \in r_b^{\mathbf{M}}$ iff $\mathbf{M} \models r_b(t_1, \ldots, t_n)[v]$. Thus, by structural induction, $\mathbf{N} \models \psi$ iff $\mathbf{M} \models \psi^\circ$. This immediately gives that for each model $\mathbf{M} \in \mathrm{Mod}(T^\circ \cup T_L^\circ)$ there is a model $\mathbf{N} \in \mathrm{Mod}(T)$ such that $\mathbf{N} \models \psi$ iff $\mathbf{M} \models \psi^\circ$ for any ψ. As a consequence, if $T \models \varphi$ then $T^\circ \cup T_L^\circ \models \varphi^\circ$. □

Note that if $\mathbf{L}$ is a general complete lattice then TBL is sound but might not be complete. Indeed, consider the following example. Let $\mathbf{L}$ be the complete lattice defined on the real unit interval $[0, 1]$ with $\wedge$ and $\vee$ being the minimum and maximum, respectively. Let $T = \{\langle r, 1 - (1/n)\rangle | n \in \mathbb{N}\}$ be a theory of a language, where r is a nullary relation symbol. Then we have $T \models \langle r, 1\rangle$ since $r^{\mathbf{M}} = 1$ for each model $\mathbf{M} \in \mathrm{Mod}(T)$. On the contrary, $T^\circ \cup T_L^\circ \not\models r_1$, because there is a model $\mathbf{M} \in \mathrm{Mod}(T^\circ \cup T_L^\circ)$ such that $\mathbf{M} \models r_a$ for each $a < 1$ and $\mathbf{M} \not\models r_1$.

5. Languages with equality

In this section we are going to propose a generalization of the equality predicate in terms of threshold Boolean logic. The section is not intended to be comprehensive, we mainly present basic ideas and give some comments. In the classical predicate logic, we usually consider a binary relation symbol $\approx$ standing for equality. Atomic formulas $t \approx t'$ are called identities. The intended meaning of $t \approx t'$ is "t equals t'". So, the aim of $\approx$ is to describe the indistinguishability of elements. Equality is interesting because it needs a special treatment: we want $\approx$ to be interpreted in each structure by the identity relation, however, it is a well-known fact that property "being an identity relation" is not axiomatizable in first-order logic. The generalization presented below is, in a sense, an extension of previous results of Bělohlávek and Vychodil (2006a,b,c).

By an $\mathbf{L}$-*language with equality* we mean an $\mathbf{L}$-language $\mathcal{L}$ which contains the binary relation symbol $\approx$. Atomic threshold formulas of the form $\langle \approx(t, t'), a\rangle$, written $\langle t \approx t', a\rangle$, will be called *graded identities*. In what follows we are going to describe desired properties $\approx$ by additional axioms of TBL. Note in advance that in fuzzy setting there might be several reasonable interpretations of $\approx$. For instance, one can require $\approx$ to express "being similar", "being (fully) equal", etc. We first show a way to axiomatize $\approx$ that is, in our opinion, minimal. Later on, we show a general method to extend the axiom schemes to get more desirable properties of $\approx$. The following axiom schemes will be our *basic axiom schemes of equality*:

$$\langle x \approx x, 1\rangle \tag{1}$$

$$(\langle x_1 \approx y_1, 1\rangle \ \&\cdots\&\ \langle x_n \approx y_n, 1\rangle \ \&\ \langle r(x_1, \ldots, x_n), 1\rangle) \rightarrow \langle r(y_1, \ldots, y_n), 1\rangle \tag{2}$$

$$(\langle x_1 \approx y_1, 1\rangle \ \&\cdots\&\ \langle x_n \approx y_n, 1\rangle) \rightarrow \langle f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n), 1\rangle, \tag{3}$$

$x, x_1, y_1, \ldots, x_n, y_n$ are variables, r is an n-ary relation symbol (including $\approx$), and f is an n-ary function symbol. It is immediate that equations (1)–(3) are straightforward generalization of the classical equality axioms. In case of languages with equality, we extend the notion of provability by assuming also axiom schemes equations (1)–(3). Then, by standard arguments, we get that

$$\langle x \approx y, 1\rangle \rightarrow \langle y \approx x, 1\rangle, \tag{4}$$

$$(\langle x \approx y, 1\rangle \ \&\ \langle y \approx z, 1\rangle) \rightarrow \langle x \approx z, 1\rangle, \tag{5}$$

are provable (from any theory).

As a consequence, for each **L**-structure which is a model of equations (1)–(3) we have that the 1-cut of $\approx^{\mathbf{M}}$ is a classical equivalence relation. Analogously as in PC, our basic requirement is that the 1-cut of $\approx^{\mathbf{M}}$ should be an identity relation. In other words, we wish elements $\mathfrak{u}$ and $\mathfrak{v}$ to be fully equal (i.e. equal in degree 1) iff they are identical (indistinguishable). With respect to this, a structure **M** for language with equality is called a *model of equality* if for each $\mathfrak{u}, \mathfrak{v} \in M$ we have $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} = 1$ iff $\mathfrak{u} = \mathfrak{v}$. Clearly, each model of equality satisfies equations (1)–(3).

The following theorem shows that TBL with equality is complete over the semantics given by models of equality.

THEOREM 4 (COMPLETENESS FOR LANGUAGES WITH EQUALITY). For each theory T of language $\mathcal{L}$ with equality and for any formula φ of that language we have $T \vdash \varphi$ iff $\mathbf{M} \models \varphi$ for each $\mathbf{M} \in \mathrm{Mod}(T)$ which is a model of equality.

Proof. Inspect the proof of Theorem 3 and observe the following. $\mathcal{L}_{\mathrm{L}}$ is a classical predicate language in which we can identify $\approx_1$ with the equality relation symbol. Throughout this proof, we call a classical structure **M** for $\mathcal{L}_{\mathrm{L}}$ a model of equality iff $\approx_1^{\mathbf{M}}$ is the identity relation on M. Clearly, $T \vdash \varphi$ iff $T^\circ \cup T_L^\circ \vdash \varphi^\circ$ iff φ° is true in each model $\mathbf{M} \in \mathrm{Mod}(T^\circ \cup T_L^\circ)$ which is a model of equality (this follows from the completeness of the classical predicate logic considering a language with equality $\approx_1$). Thus, it remains to show that

φ° is true in each $\mathbf{M} \in \mathrm{Mod}(T^\circ \cup T_L^\circ)$ which is a model of equality iff
φ is true in each $\mathbf{M} \in \mathrm{Mod}(T)$ which is a model of equality.

But this is easy to see. For the "$\Rightarrow$"-part it suffices to show that for each $\mathbf{M} \in \mathrm{Mod}(T)$ being a model of equality, there is a classical model $\mathbf{M}^\circ \in \mathrm{Mod}(T^\circ \cup T_L^\circ)$ of equality such that for any ψ we have $\mathbf{M} \models \psi$ iff $\mathbf{M}^\circ \models \psi^\circ$. Obviously, $\mathbf{M}^\circ$ introduced in the proof of Theorem 3 does the job (recall that the 1-cut of $\approx^{\mathbf{M}}$ is the identity on M). The same idea applies also for the "$\Leftarrow$"-part. For $\mathbf{M} \in \mathrm{Mod}(T^\circ \cup T_L^\circ)$ we define **N** as in the proof of Theorem 3. Here we can use the observation that if $\mathfrak{u} \approx_1^{\mathbf{M}} \mathfrak{v}$ then $\mathfrak{u} \approx^{\mathbf{N}} \mathfrak{v} = 1$; and conversely if $\mathfrak{u} \approx^{\mathbf{N}} \mathfrak{v} = 1$ then $\mathfrak{u} \approx_1^{\mathbf{M}} \mathfrak{v}$ by Lemma 2 (iii). □

Even if axioms (1)–(3) ensure that the 1-cut of $\approx^{\mathbf{M}}$ is a classical equivalence relation which is compatible with functions and 1-cuts of **L**-relations $r^{\mathbf{M}}$, these axioms do not influence a-cuts of $\approx^{\mathbf{M}}$ for $a \neq 1$. From this point of view, our basic equality axioms are quite modest. In the sequel we demonstrate a way to extend these axioms and thus to specify further properties of $\approx^{\mathbf{M}}$.

For instance, in fuzzy setting it is convenient to have $\approx^{\mathbf{M}}$ interpreted by a *compatible similarity relation*. Similarity relations are particular (binary) **L**-relations which serve as interpretations of "being similar". The (graded) compatibility ensures that pairwise similar arguments, if supplied to functions of $F^{\mathbf{M}}$, yield pairwise similar results. Furthermore, compatibility yields that if some elements are $r^{\mathbf{M}}$-related then also pairwise similar elements are $r^{\mathbf{M}}$-related. So, from the point of view of fuzzy logic, compatible similarity relations are natural ones which enables us to formalize "similarity of elements in the universe of discourse" and the fact that "similar elements would behave similarly".

In order to define a suitable notion of compatibility, we introduce additional operations on Noetherian lattices so that they become (Noetherian) *residuated lattices*. Residuated lattices, being initially developed in the 1930s by Dilworth (1938) and Ward and Dilworth (1939) in ring theory, were introduced into the context of fuzzy logic by Goguen (1967, 1968, 1969). More details on the role of residuated lattices in fuzzy logic and fuzzy relational systems can be obtained from monographs Hájek (1998), Gottwald (2001), Bělohlávek (2002a,b). Recall that a residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \Rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice,
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid,
(iii) $a \otimes b \leq c$ iff $a \leq b \Rightarrow c$ is true for each $a, b, c \in L$.

In fuzzy logic, $\otimes$ and $\Rightarrow$ are usually used as truth functions (interpretations) of "fuzzy conjunction" and "fuzzy implication", respectively. A residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \Rightarrow, 0, 1\rangle$ is called *Noetherian residuated lattice* if $\langle L, \wedge, \vee, 0, 1\rangle$ is a Noetherian lattice.

Given a Noetherian residuated lattice **L**, we consider the following axiom schemes:

(Ref) $\langle x \approx x, 1\rangle$,
(C_r) $(\langle x_1 \approx y_1, a_1\rangle \,\&\ldots\&\, \langle x_n \approx y_n, a_n\rangle \,\&\, \langle r(x_1, \ldots, x_n), b\rangle) \rightarrow \langle r(y_1, \ldots, y_n), a_1 \otimes \cdots \otimes a_n \otimes b\rangle$,
(C_f) $(\langle x_1 \approx y_1, a_1\rangle \& \cdots \& \langle x_n \approx y_n, a_n\rangle) \rightarrow \langle f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n), a_1 \otimes \cdots \otimes a_n\rangle$

where $a_1, \ldots, a_n, b \in L$, $x, x_1, y_1, \ldots, x_n, y_n$ are variables, r is an n-ary relation symbol (including $\approx$), and f is an n-ary function symbol. Evidently, (C_r) and (C_f) are schemes introducing the above-described compatibility with relations and functions. Analogously as for the basic axiom schemes, one can show that

(Sym) $\langle x \approx y, a\rangle \rightarrow \langle y \approx x, a\rangle$,
(Tra) $(\langle x \approx y, a\rangle \,\&\, \langle y \approx z, b\rangle) \rightarrow \langle x \approx z, a \otimes b\rangle$.

are provable when considering (Ref)–(C_f) as additional axiom schemes of TBL. Observe that axioms (1)–(3) are instances of (Ref)–(C_f). Thus, each model satisfying (Ref)–(C_f) satisfies also axioms (1)–(3). On the contrary, there are models of equality, and thus models of axioms (1)–(3), which do not satisfy (Ref)–(C_f).

If we add (Ref)–(C_f) to TBL as additional axiom schemes, Theorem 4 then yields that for any theory T of a language with equality we have $T \vdash \varphi$ iff $\mathbf{M} \models \varphi$ for each model of equality $\mathbf{M} \in \mathrm{Mod}(T)$ such that (Ref)–(C_f) are valid in $\mathbf{M}$. Therefore, TBL extended by (Ref)–(C_f) is syntactico-semantically complete over the semantics given by models of equality satisfying (Ref)–(C_f). This observation will be further used in Section 6.

The following assertion says that if $\mathbf{M}$ is a model of equality satisfying (Ref)–(C_f) then $\approx^{\mathbf{M}}$ is a similarity relation which is compatible with all functions and relations from $\mathbf{M}$, see Bělohlávek (2002b). Thus, if $R = \{ \approx \}$ then $\mathbf{M} = \langle M, \{ \approx^{\mathbf{M}} \}, F^{\mathbf{M}} \rangle$ is an algebra with fuzzy equality, see Bělohlávek and Vychodil (2006a,b,c).

THEOREM 5. Let $\mathbf{M} = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ be a model of equality satisfying (Ref)–(C_f). Then

(i) $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} = 1$ iff $\mathfrak{u} = \mathfrak{v}$,
(ii) $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} = \mathfrak{v} \approx^{\mathbf{M}} \mathfrak{u}$,
(iii) $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} \otimes \mathfrak{v} \approx^{\mathbf{M}} \mathfrak{w} \leq \mathfrak{u} \approx^{\mathbf{M}} \mathfrak{w}$,
(iv) $\mathfrak{u}_1 \approx^{\mathbf{M}} \mathfrak{v}_1 \otimes \cdots \otimes \mathfrak{u}_n \approx^{\mathbf{M}} \mathfrak{v}_n \otimes r^{\mathbf{M}}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \leq r^{\mathbf{M}}(\mathfrak{v}_1, \ldots, \mathfrak{v}_n)$,
(v) $\mathfrak{u}_1 \approx^{\mathbf{M}} \mathfrak{v}_1 \otimes \cdots \otimes \mathfrak{u}_n \approx^{\mathbf{M}} \mathfrak{v}_n \leq f^{\mathbf{M}}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \approx^{\mathbf{M}} f^{\mathbf{M}}(\mathfrak{v}_1, \ldots, \mathfrak{v}_n)$,

where $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{u}_1, \mathfrak{v}_1, \ldots, \mathfrak{u}_n, \mathfrak{v}_n \in M$, $r^{\mathbf{M}} \in R^{\mathbf{M}}$ is n-ary, and $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is n-ary.

Proof. The proof is easy. For example, we show (iii). Since $\mathbf{M}$ is a model of (Tra), we have that $\mathbf{M} \models \langle x \approx y, a \rangle[v]$ and $\mathbf{M} \models \langle y \approx z, b \rangle[v]$ implies $\mathbf{M} \models \langle x \approx z, a \otimes b \rangle[v]$ for each $\mathbf{M}$-valuation v. Put $v(x) = \mathfrak{u}$, $v(y) = \mathfrak{v}$, and $v(z) = \mathfrak{w}$. In this particular case, $\|x\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|y\|_{\mathbf{M},v} \geq a$ and $\|y\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|z\|_{\mathbf{M},v} \geq b$ implies $\|x\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|z\|_{\mathbf{M},v} \geq a \otimes b$. That is, $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} \geq a$ and $\mathfrak{v} \approx^{\mathbf{M}} \mathfrak{w} \geq b$ implies $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{w} \geq a \otimes b$. Thus, for $a = \mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v}$ and $b = \mathfrak{v} \approx^{\mathbf{M}} \mathfrak{w}$, we have $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{w} \geq a \otimes b = \mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} \otimes \mathfrak{v} \approx^{\mathbf{M}} \mathfrak{w}$. □

Remarks. Properties of $\approx^{\mathbf{M}}$ depend on the chosen $\otimes$. For instance, if $\otimes$ coincides with $\wedge$ then (Ref)–(C_f) simplify as follows:

$$\langle x \approx x, a \rangle,$$

$$(\langle x_1 \approx y_1, a \rangle \& \cdots \& \langle x_n \approx y_n, a \rangle \& \langle r(x_1, \ldots, x_n), a \rangle) \rightarrow \langle r(y_1, \ldots, y_n), a \rangle,$$

$$(\langle x_1 \approx y_1, a \rangle \& \cdots \& \langle x_n \approx y_n, a \rangle) \rightarrow \langle f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n), a \rangle.$$

In case of $\otimes = \wedge$, each a-cut of $\approx^{\mathbf{M}}$ (i.e. not only the 1-cut) is a classical equivalence relation which is a congruence on the algebra $\langle M, F^{\mathbf{M}} \rangle$, see Bělohlávek and Vychodil (2006a,b,c).

Sometimes it is desirable to have $\approx^{\mathbf{M}}$ crisp (i.e. $\{\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v} | \mathfrak{u}, \mathfrak{v} \in M\} \subseteq \{0, 1\}$). To ensure this, we can consider the following "crispness axiom scheme":

$$\langle x \approx y, a \rangle \rightarrow \langle x \approx y, 1 \rangle \text{ if } \quad a \neq 0,$$

which gives that each $\mathfrak{u} \approx^{\mathbf{M}} \mathfrak{v}$ either is 0 or 1. Using the same ideas as above, one can show that TBL extended by this axiom scheme is complete over semantics given by models of equality with crisp interpretations of $\approx^{\mathbf{M}}$.

Let us close this section with a remark on a possible interpretation of threshold Boolean logic. TBL can be seen as a calculus for reasoning about the consequence (entailment) from classes of fuzzy structures which unlike the common fuzzy logical calculi uses two-valued logical connectives. The present section showed that truth functions of "fuzzy logical connectives" are still useful even if the studied entailment relations are bivalent—we can have truth functions of fuzzy logical connectives as reasonable "parameters of compatibility". From this point of view, TBL is a logical framework for reasoning about fuzzy structures with compatible similarity relations. In an analogous way one can study other properties of fuzzy relations and fuzzy structures inside TBL.

6. Connection to fuzzy Horn logic

The aim of this section is to show that in certain cases the completeness of fuzzy Horn logic can be easily obtained from the completeness of TBL. The subsequent result can be seen as a particular example of how TBL can be used to prove a claim about special fuzzy structures and their properties.

In the initial papers on fuzzy Horn logic (FHL), see Bělohlávek and Vychodil (2006a,b,c), we introduced a syntactico-semantically complete calculus for reasoning with particular implications between identities (so-called Horn clauses with truth-weighted premises). FHL is developed in Pavelka-style, i.e. we used a fixed complete residuated lattice as the structure of truth degrees, we defined a degree of provability, and the completeness theorem put in equilibrium provability degrees with degrees of semantic entailment. The relationship between FHL and TBL might be interesting from several viewpoints. First, both FHL and TBL are based on the idea of truth-evaluated syntax. Second, formulas of FHL have truth-weighted premises which can be seen as particular threshold formulas. Third, FHL operates with provability degrees, however, TBL does not (a formula either is provable or not). Realizing that, one can ask whether we can prove Pavelka-style completeness of FHL by means of the completeness of TBL. We are going to show that the answer is positive.

First, we summarize basic notions of fuzzy Horn logic. Let $\mathbf{L}$ be a Noetherian residuated lattice, $\mathcal{L}$ be a language with equality. A *Horn clause* (*of language* $\mathcal{L}$) is an expression of the form

$$(*) \quad (\langle s_1 \approx s'_1, a_1\rangle \,\&\cdots\&\, \langle s_n \approx s'_n, a_n\rangle) \longrightarrow t \approx t',$$

where $s_1, s'_1, \ldots, s_n, s'_n, t, t'$ are terms of $\mathcal{L}$, and $a_1, \ldots, a_n \in L$.

A Horn clause of the form (*) will be abbreviated by $P \longrightarrow t \approx t'$, where P stands for $\langle s_1 \approx s'_1, a_1\rangle \,\&\cdots\&\, \langle s_n \approx s'_n, a_n\rangle$. Note that P is a threshold formula, because it is a conjunction of atomic threshold formulas. However, Horn clause $P \longrightarrow t \approx t'$ is *not* a formula of TBL ($t \approx t'$ is a classical atomic formula which is not a threshold formula). **L**-sets of Horn clauses (of the same language) will be denoted by $\Sigma, \Sigma', \ldots, \Sigma(P \longrightarrow t \approx t')$ being interpreted as "the degree to which $P \longrightarrow t \approx t'$ belongs to Σ". For any **L**-set Σ of Horn clauses of language $\mathcal{L}$ one can consider a theory $\Sigma^{\bullet}$ of threshold formulas (of $\mathcal{L}$) such that

$$\Sigma^{\bullet} = \{P \longrightarrow \langle t \approx t', \Sigma(P \longrightarrow t \approx t')\rangle \mid P \longrightarrow t \approx t' \text{ is a Horn clause of } \mathcal{L}\}$$

Each $P \rightarrow \langle t \approx t', \Sigma(P \rightarrow t \approx t')\rangle$ is a formula of TBL because $\Sigma(P \rightarrow t \approx t') \in L$, i.e. $\langle t \approx t', \Sigma(P \rightarrow t \approx t')\rangle$ is an atomic threshold formula.

We now introduce basic notions of the semantics of FHL. Let $P \rightarrow t \approx t'$ be a Horn clause of language $\mathcal{L}$. Given an **L**-structure for $\mathcal{L}$ and an **M**-valuation v, we define the *degree* $\|P \rightarrow t \approx t'\|_{\mathbf{M},v}$ to which $P \rightarrow t \approx t'$ *of the form* (*) *is true* in **M** *under* v as follows:

$$\|P \rightarrow t \approx t'\|_{\mathbf{M},v} = \begin{cases} \|t\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t'\|_{\mathbf{M},v} & \text{if } a_i \leq \|s_i\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|s_i'\|_{\mathbf{M},v} \\ & \text{for each } i = 1, \ldots, n, \\ 1 & \text{otherwise.} \end{cases}$$

The degree $\|P \rightarrow t \approx t'\|_{\mathbf{M}}$ *to which* $P \rightarrow t \approx t'$ *is true in* **M** is defined using the infimum which ranges over all **M**-valuations:

$$\|P \rightarrow t \approx t'\|_{\mathbf{M}} = \bigwedge \{\|P \rightarrow t \approx t'\|_{\mathbf{M},v} | v \text{ is an } \mathbf{M} - \text{valuation}\}.$$

Let Σ be an **L**-set of Horn clauses of $\mathcal{L}$. An **L**-structure **M** for $\mathcal{L}$ is called a *model of* Σ, written $\mathbf{M} \in \mathrm{Mod}(\Sigma)$, if **M** is a model of equality (see Section 5) which satisfies (Ref)–(C_f), and $\Sigma(P \rightarrow t \approx t') \leq \|P \rightarrow t \approx t'\|_{\mathbf{M}}$ for each Horn clause $P \rightarrow t \approx t'$. A *degree* $\|P \rightarrow t \approx t'\|_{\Sigma}$ *to which* $P \rightarrow t \approx t'$ *semantically follows from* Σ is defined by

$$\|P \rightarrow t \approx t'\|_{\Sigma} = \bigwedge \{\|P \rightarrow t \approx t'\|_{\mathbf{M}} | \mathbf{M} \in \mathrm{Mod}(\Sigma)\}.$$

For brevity, we assume that the model class $\mathrm{Mod}(\Sigma^\bullet)$ contains only the models of $\Sigma^\bullet$ which are models of equality and satisfy (Ref)–(C_f).

LEMMA 6. Let **M** be an **L**-structure for a language with equality. Then

(i) $a \leq \|P \rightarrow t \approx t'\|_{\mathbf{M}}$ iff $\mathbf{M} \models P \rightarrow \langle t \approx t', a\rangle$,
(ii) $\mathrm{Mod}(\Sigma) = \mathrm{Mod}(\Sigma^\bullet)$ for any **L**-set Σ of Horn clauses.

Proof. (i): Suppose $P \rightarrow t \approx t'$ is an abbreviation for (*). $a \leq \|P \rightarrow t \approx t'\|_{\mathbf{M}}$ is true iff for each **M**-valuation v we have $a \leq \|P \rightarrow t \approx t'\|_{\mathbf{M},v}$ iff for each v we have

$$\text{IF } a_i \leq \|s_i\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|s_i'\|_{\mathbf{M},v} \text{ for each } i = 1, \ldots n,$$
$$\text{THEN } a \leq \|t\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t'\|_{\mathbf{M},v},$$

which is true iff for each v we have $\mathbf{M} \models P \rightarrow \langle t \approx t', a\rangle[v]$ iff $\mathbf{M} \models P \rightarrow \langle t \approx t', a\rangle$.

(ii): Let **M** be a model of equality which satisfies (Ref)–(C_f). We have $\mathbf{M} \in \mathrm{Mod}(\Sigma)$ iff $\Sigma(P \rightarrow t \approx t') \leq \|P \rightarrow t \approx t'\|_{\mathbf{M}}$ for each Horn clause $P \rightarrow t \approx t'$. By (i), this is equivalent to $\mathbf{M} \models P \rightarrow \langle t \approx t', \Sigma(P \rightarrow t \approx t')\rangle$ for each $P \rightarrow t \approx t'$ which is true iff $\mathbf{M} \in \mathrm{Mod}(\Sigma^\bullet)$. □

For any Horn clause $P \rightarrow t \approx t'$ and any **L**-set Σ of Horn clauses we define the *degree* $|P \rightarrow t \approx t'|_{\Sigma}$ *to which* $P \rightarrow t \approx t'$ is *provable from* Σ by

$$|P \rightarrow t \approx t'|_{\Sigma} = \bigvee \{a \in L | \Sigma^\bullet \vdash P \rightarrow \langle t \approx t', a\rangle\},$$

where $\vdash$ is the provability in TBL extended by assuming axiom schemes (Ref)–(C_f). We now have the following characterization.

THEOREM 7. (COMPLETENESS of FHL). Let $\mathbf{L}$ be a Noetherian residuated lattice and let Σ be an $\mathbf{L}$-set of Horn clauses of language $\mathcal{L}$. Then for each Horn clause $P \to t \approx t'$ of $\mathcal{L}$ we have $|P \to t \approx t'|_\Sigma = \|P \to t \approx t'\|_\Sigma$.

Proof. Taking into account Theorem 4 and Lemma 6, we get

$$\|P \to t \approx t'\|_\Sigma = \bigvee \{a \in L | a \leq \|P \to t \approx t'\|_{\mathbf{M}} \text{ for each } \mathbf{M} \in \mathrm{Mod}(\Sigma)\} = \bigvee \{a \in L | \mathbf{M} \models P \to \langle t \approx t', a\rangle \text{ for each } \mathbf{M} \in \mathrm{Mod}(\Sigma^\bullet)\} = \bigvee \{a \in L | \Sigma^\bullet \vdash P \to \langle t \approx t', a\rangle\} = |P \to t \approx t'|_\Sigma,$$

which is the desired equality. □

Remarks. Described verbally, Theorem 7 says that the degree to which $P \to t \approx t'$ semantically follows from Σ equals to the supremum of truth degrees $a \in L$ such that $P \to \langle t \approx t', a\rangle$ is provable from $\Sigma^\bullet$. Hence, $\Sigma^\bullet \vdash P \to \langle t \approx t', a\rangle$ provides us with a lower estimation $a \in L$ of the degree $\|P \to t \approx t'\|_\Sigma$. Moreover, the fact that $\mathbf{L}$ is a Noetherian lattice together with (Lea) and (Sup) gives $\|P \to t \approx t'\|_\Sigma = a$ iff a is the greatest truth degree such that $\Sigma^\bullet \vdash P \to \langle t \approx t', a\rangle$. So, even if $\mathbf{L}$ can be an infinite structure of truth degrees, the notion of a degree of provability is finitary in the sense that we cannot have a situation where $|\ldots|_\Sigma$ is strictly greater than all estimations of $|\ldots|_\Sigma$ given by (finite) proofs. This is an important distinction from the Pavelka-style predicate fuzzy logic which uses the standard Łukasiewicz algebra as the structure of truth degrees, see Gerla (2001), Hájek (1998), Novák *et al.* (1999).

A note for readers familiar with Bělohlávek and Vychodil (2006a,b,c): the completeness given by Theorem 7 applies to the interpretation of Horn clauses which is determined by the so-called globalization, see Takeuti and Titani (1987). Theorem 7 does not cover the other interpretations. Furthermore, Theorem 7 operates with Horn clauses with unrestricted premises and $\mathbf{L}$ is supposed to be a Noetherian lattice. From this point of view, Theorem 7 applies only to a particular fragment of FHL. In addition to that, unlike the results in Bělohlávek and Vychodil (2006a,b,c), Theorem 7 does not provide us with a simple deductive system which enables us to infer "weighted Horn clauses from weighted Horn clauses". Each elementary proof step in the original deductive system of FHL is now corresponding to several proof steps in TBL which might be not very transparent.

7. Characterization of elementary classes

So far, the research in fuzzy logic has been focused almost exclusively on the aspects motivated by the proof theory. Not much attention has been paid to the structural properties of fuzzy structures and the characterization of model classes by closure properties. As an exception, in Gottwald (2001) the author presents an approach to ultraproducts in fuzzy setting which is based on the ideas of the so-called continuous model theory, see Chang and Keisler (1966). An analogous result of Ying (1992) deals with ultraproducts in the context of Pavelka-style fuzzy logic. Various closure properties of fuzzy structures of the equational fragment of fuzzy logic were studied in Bělohlávek (2003), Bělohlávek and Vychodil (2006a,b,c). In the sequel we show an alternative approach to ultraproducts in fuzzy setting and show its nontrivial application for the characterization of elementary classes of fuzzy structures.

First, we introduce a suitable generalization of ultraproducts of **L**-structures. Let $\{\mathbf{M}_i | i \in I\}$ be a system of **L**-structures for language $\mathcal{L}$. Denote the Cartesian product of $\{\mathbf{M}_i | i \in I\}$ by $\Pi_{i\in I} M_i$. That is, $\Pi_{i\in I} M_i$ is a set of all mappings $u : I \to \bigcup_{i\in I} M_i$ such that $\mathrm{u}(i) \in M_i (i \in I)$. Let W be an ultrafilter over I, i.e. W is a subset of the power set of I satisfying

(i) if $X, Y \in W$ then $X \cap Y \in W$,
(ii) if $X \in W$ and $X \subseteq Y$ then $Y \in W$,
(iii) for each $X \subseteq I$ we have $X \in W$ iff $I - X \notin W$.

We define a (classical) binary relation θ_W on $\Pi_{i\in I} M_i$ by

$$\langle \mathrm{u}, \mathfrak{v}\rangle \in \theta_W \text{ iff } \{i \in I | \mathrm{u}(i) = \mathfrak{v}(i)\} \in W.$$

Since θ_W is defined the same way as for the classical structures, we immediately obtain that θ_W is a classical equivalence relation. Let $[\mathrm{u}]_W$ denote the equivalence class of θ_W which contains u, i.e. $[\mathrm{u}]_W = \{\mathfrak{v} | \langle \mathrm{u}, \mathfrak{v}\rangle \in \theta_W\}$. Furthermore, we consider **L**-structure $\Pi_W \mathbf{M}_i$ whose universe set is the factor set $(\Pi_{i\in I} M_i)/\theta_W$; operations $f^{\Pi_W \mathbf{M}_i} \in F^{\Pi_W \mathbf{M}_i}$ are defined by

$$f^{\Pi_W \mathbf{M}_i}([\mathrm{u}_1]_W, \ldots, [\mathrm{u}_n]_W) = \left[f^{\Pi_{i\in I}\mathbf{M}_i}(\mathrm{u}_1, \ldots, \mathrm{u}_n)\right]_W,$$

where $f^{\Pi_{i\in I}\mathbf{M}_i}(\mathrm{u}_1, \ldots, \mathrm{u}_n)(i) = f^{\mathbf{M}_i}(\mathrm{u}_1(i), \ldots, \mathrm{u}_n(i))$ $(i \in I)$; for each n-ary relation symbol $r \in R$ and for any $[\mathrm{u}_1]_W, \ldots, [\mathrm{u}_n]_W$ we put

$$r^{\Pi_W M_i}([\mathrm{u}_1]_W, \ldots, [\mathrm{u}_n]_W) = \bigvee \{a \in L | \{i \in I | r^{M_i}(\mathrm{u}_1(i), \ldots, \mathrm{u}_n(i)) \geq a\} \in W\}$$

It can be shown that $\Pi_W \mathbf{M}_i$ is a well-defined **L**-structure. First, note that the definition of the functional part of $\Pi_W \mathbf{M}_i$ is exactly the same as in the case of classical structures. Furthermore, we show that each $r^{\Pi_W \mathbf{M}_i} \in R^{\Pi_W \mathbf{M}_i}$ is a well-defined **L**-relation. It suffices to check that $r^{\Pi_W \mathbf{M}_i}([\mathrm{u}_1]_W, \ldots, [\mathrm{u}_n]_W)$ does not depend on the elements chosen from classes $[\ldots]_W$: take $\mathfrak{v}_j \in [\mathrm{u}_j]_W$ for each $j = 1, \ldots, n$ and let $\{i \in I | r^{\mathbf{M}_i}(\mathrm{u}_1(i), \ldots, \mathrm{u}_n(i)) \geq a\} \in W$. Then

$$\left\{i \in I | r^{\mathbf{M}_i}(\mathrm{u}_1(i), \ldots, \mathrm{u}_n(i)) \geq a\right\} \cap \bigcap_{j=1}^{n} \{i \in I | \mathrm{u}_j(i) = \mathfrak{v}_j(i)\} \in W.$$

Thus, we have

$$\{i \in I | r^{\mathbf{M}_i}(\mathrm{u}_1(i), \ldots, \mathrm{u}_n(i)) \geq a \text{ and } \mathrm{u}_j(i) = \mathfrak{v}_j(i) \text{ for } \quad j = 1, \ldots, n\} \in W.$$

Therefore, $\{i \in I | r^{\mathbf{M}_i}(\mathfrak{v}_1(i), \ldots, \mathfrak{v}_n(i)) \geq a\} \in W$. Hence,

$$r^{\Pi_W \mathbf{M}_i}([\mathrm{u}_1]_W, \ldots, [\mathrm{u}_n]_W) = r^{\Pi_W \mathbf{M}_i}([\mathfrak{v}_1]_W, \ldots, [\mathfrak{v}_n]_W).$$

To sum up, $\Pi_W \mathbf{M}_i$ is a well-defined **L**-structure. In what follows, $\Pi_W \mathbf{M}_i$ will be called the *ultraproduct of* $\{\mathbf{M}_i | i \in I\}$ *modulo W.*