
Fuzzy Closure Operators with Truth Stressers

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Abstract

We study closure operators and closure structures in a fuzzy setting. Our main interest is the monotony condition of closure operators. In a fuzzy setting, the monotony condition may take several particular forms, all of them equivalent in the bivalent case. We study closure operators, called fuzzy closure operators with truth stresser, satisfying the monotony condition which can be linguistically described as “if it is (very) true that A is included in B then the closure of A is included in the closure of B .” We present examples of closure operators with truth stresser, investigate their basic properties and related structures.

Keywords: closure operator, closure system, Galois connection, fuzzy set, residuated lattice

1 Introduction and preliminaries

Closure and interior operators on ordinary sets belong to fundamental mathematical structures with direct applications, both in mathematics (topology, logic, for instance) and other areas (e.g. data mining, knowledge representation, deductive reasoning). In fuzzy set theory, both general closure operators which operate with fuzzy sets (so called fuzzy closure operators) and several particular closure operators were studied, e.g. operators in fuzzy topology, consequence operators in fuzzy logic, formal concept formation operators in formal concept analysis, operators induced by binary fuzzy relations, see e.g. [2, 3, 8, 9, 13, 17, 18].

The point of our interest is the monotony condition. In the bivalent case, monotony of a closure operator C says that if $A \subseteq B$ then $C(A) \subseteq C(B)$. In earlier studies, monotony for fuzzy closure operators meant just this condition with A and B being fuzzy sets and $A \subseteq B$ meaning that $A(x) \leq B(x)$ for each element x of the universe set. As shown e.g. in [3], several fuzzy closure operators satisfy stronger conditions of monotony which enable us to tell more about the closure operator. For instance, one can obtain generalizations of theorems from bivalent case for which the above weaker monotony is not sufficient, one can form a factor structure of the set of all fixed points (which is a rough version of the whole set of all fixed points and has applications in formal concept analysis), etc. In the present

paper, we develop an approach to fuzzy closure operators based on an interpretation of monotony saying “if it is very true that A is included in B then $C(A)$ is included in $C(B)$ ”. This approach generalizes some earlier approaches in that the weak form of monotony and the strongest (in a sense) form of monotony are its particular cases.

In the rest of this section, we present preliminaries on fuzzy logic and fuzzy sets. We use complete residuated lattices as structures of truth degrees, i.e. structures $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with the least element 0 and the greatest element 1; $\langle L, \otimes, 1 \rangle$ is a commutative monoid, i.e. \otimes is commutative, associative, and $x \otimes 1 = x$ holds for each $x \in L$; \otimes and \rightarrow form an adjoint pair, i.e. $x \otimes y \leq z$ iff $x \leq y \rightarrow z$ holds for all $x, y, z \in L$. Residuated lattices have been introduced into the context of fuzzy logic by Goguen [14]. We refer e.g. to [3, 15] for further information. Recall that the most studied and applied residuated lattices are those defined on the real interval $[0, 1]$ (residuated lattices on $[0, 1]$ uniquely correspond to left-continuous t-norms). The three most important pairs of adjoint operations are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). More generally, $\langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1 \rangle$ is a complete residuated lattice iff \otimes is a left-continuous t-norm and $a \rightarrow b = \bigvee \{z \mid a \otimes z \leq b\}$. Another important case is $L = \{a_0 = 0, a_1, \dots, a_n = 1\}$ ($a_0 < \dots < a_n$) with \otimes given by $a_k \otimes a_l = a_{\max(k+l-n, 0)}$ and the corresponding \rightarrow given by $a_k \rightarrow a_l = a_{\min(n-k+l, n)}$, or \otimes and \rightarrow being the restrictions of Gödel operations from $[0, 1]$ to L . A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$. A biresiduum of a and b is denoted by $a \leftrightarrow b$ and is defined by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$.

An \mathbf{L} -set (fuzzy set) [14] A in a universe set X is any map $A : X \rightarrow L$. By L^X we denote the set of all \mathbf{L} -sets in X . The concept of an \mathbf{L} -relation is defined obviously. Operations on L extend pointwise to L^X , e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for $A, B \in L^X$. Following common usage, we write $A \cup B$ instead of $A \vee B$, etc. Given $A, B \in L^X$, the subethood degree $S(A, B)$ of A in B is defined by $S(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x)$. We write $A \subseteq B$ if $S(A, B) = 1$. Analogously, the equality degree $E(A, B)$ of A and B is defined by $E(A, B) = \bigwedge_{x \in X} A(x) \leftrightarrow B(x)$. It is immediate that $E(A, B) = S(A, B) \wedge S(B, A)$. By $\{a_1/x_1, \dots, a_n/x_n\}$ we denote an \mathbf{L} -set A for which $A(x) = a_i$ if $x = x_i$ ($i = 1, \dots, n$) and $A(x) = 0$ otherwise. By \emptyset and X we denote the empty and full \mathbf{L} -set in X , i.e. $\emptyset(x) = 0$ and $X(x) = 1$ for each $x \in X$.

2 Fuzzy closure operators with truth stressers

As discussed in Section 1, our aim is to focus on the monotony condition of closure operators in a fuzzy setting. In particular, we are going to propose a condition which reads that for fuzzy sets A and B , “if it is (very) true that A is a subset of B then the closure of A is a subset of the closure of B ”. Since we are using residuated implication \rightarrow , the fact that $a \rightarrow b = 1$ is equivalent to $a \leq b$ means that we, in fact, require that the truth degree of “it is (very) true that A is a subset of B ” is less or equal to the truth degree of “the closure of A is a subset of the closure of B ”. In this respect, it remains to set the meaning of “ A is a subset of B ” and “(very) true”. First, for “ A is a subset of B ” we use the well-known degree $S(A, B)$ (see Section 1), i.e. the proposition “ A is a subset of B ” is assigned a degree $S(A, B) \in L$. Note that $S(A, B)$ can be verbally described as the degree to which it is true that for each $x \in X$, if x belongs to A then x belongs to B . Second, for “(very) true” we use the concept of a so-called truth stresser. Truth stressers are examples of what are called hedges in fuzzy

logic in the broader sense. Truth stressers, being probably first mentioned in [19] under the term globalization, were introduced to (mathematical) fuzzy logic by Baaz [1] and gained a considerable interest recently, see e.g. [15, 16]. Basically, a truth stresser is a unary function $*$ on L sending $a \in L$ to $a^* \in L$ which interprets connectives like “true”, “very true”, “very very true”, etc. That is, if $\|\cdot\|$ denotes the truth degree of \cdot , then for a proposition φ , the truth degree $\|\varphi$ is (very) true $\|$ of “ φ is (very) true” is $\|\varphi\|^*$. Several properties of truth stressers were described in the literature. The most important for us are the following ones:

$$1^* = 1, \tag{2.1}$$

$$a^* \leq a, \tag{2.2}$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*. \tag{2.3}$$

Another property we will use is

$$a^* = a^{**}. \tag{2.4}$$

(2.1)–(2.3) have some consequences, among which we will use

$$a \leq b \text{ implies } a^* \leq b^*, \tag{2.5}$$

$$a^* \otimes b^* \leq (a \otimes b)^*. \tag{2.6}$$

Indeed, if $a \leq b$ then $a \rightarrow b = 1$, and so $1 = (a \rightarrow b)^* \leq a^* \rightarrow b^*$ by (2.1) and (2.3), from which it follows $a^* \leq b^*$ proving (2.5). $a^* \otimes b^* \leq (a \otimes b)^*$ holds true iff $a^* \leq b^* \rightarrow (a \otimes b)^*$. Now, since $a \leq b \rightarrow (a \otimes b)$, (2.5) and (2.3) give $a^* \leq (b \rightarrow (a \otimes b))^* \leq b^* \rightarrow (a \otimes b)^*$ proving (2.6).

Note that (2.1)–(2.6) have a natural interpretation. For instance, (2.1) says that if the truth degree of φ is 1 (full truth) then the truth degree of “ φ is very true” is 1 as well. (2.5) says that if the truth degree of φ is less or equal to the truth degree of ψ , then the truth degree of “ φ is very true” is less or equal to the truth degree of “ ψ is very true”, etc.

In what follows, \mathbf{L}^* denotes a complete residuated lattice equipped with a unary function $*$: $L \rightarrow L$ satisfying (2.1)–(2.3).

EXAMPLE 2.1

Let \mathbf{L} be a complete residuated lattice.

(a) Let $*$ be the identity on L , i.e.

$$a^* = a, \tag{2.7}$$

$a \in L$. Then $*$ satisfies (2.1)–(2.3). $*$ may be thought of as interpreting the connective “ \dots is true”.

(b) Let $*$ be defined by

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Then $*$, called a globalization in [19], satisfies (2.1)–(2.3). $*$ may be thought of as interpreting the connective “ \dots is fully true”.

(c) Denote the two above truth stressers by $*^1$ (identity) and $*^2$ (globalization). Trivially, for each truth stresser $*$ we have $a^{*^2} = 0 \leq a^* \leq a = a^{*^1}$ for $a < 1$ and $1 = 1^{*^1} = 1^* = 1^{*^2}$. Therefore, truth stressers are bounded by $*^1$ and $*^2$.

(d) For any \mathbf{L} , $c_1, \dots, c_k \in L$, and nonnegative integers n_1, \dots, n_k , let

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ \bigvee_{i=1}^k c_i \otimes a^{n_i} & \text{if } a < 1, \end{cases} \tag{2.9}$$

where a^{n_i} is $a \otimes \dots \otimes a$ (n_i -times). Then $*$ is a truth stresser. Indeed, (2.1) and (2.2) are obvious. For (2.3), distinguish the following cases. First, $a = 1$, then (2.3) becomes $b^* \leq b^*$ which is true. Second, $1 \neq a \leq b$, then (2.3) is equivalent to $a^* \leq b^*$ which is true for $b = 1$ (since $a^* \leq 1$) as well as for $b < 1$ (by monotony of \otimes). Third, $1 \neq a \not\leq b \neq 1$, then (2.3) becomes $\bigvee_{i=1}^k c_i \otimes (a \rightarrow b)^{n_i} \leq \bigvee_{i=1}^k c_i \otimes a^{n_i} \rightarrow \bigvee_{i=1}^k c_i \otimes b^{n_i}$. By adjointness, this inequality is true iff for each p, q we have $c_p \otimes (a \rightarrow b)^{n_p} \otimes c_q \otimes a^{n_q} \leq \bigvee_{i=1}^k c_i \otimes b^{n_i}$ which is true. Indeed, for $n_p \leq n_q$, $c_p \otimes (a \rightarrow b)^{n_p} \otimes c_q \otimes a^{n_q} \leq c_p \otimes (a \rightarrow b)^{n_p} \otimes c_q \otimes a^{n_p} \leq c_p \otimes c_q \otimes a^{n_p} \otimes (a \rightarrow b)^{n_p} \leq c_p \otimes c_q \otimes b^{n_p} \leq c_p \otimes b^{n_p} \leq \bigvee_{i=1}^k c_i \otimes b^{n_i}$, and analogously for $n_p > n_q$.

(e) Consider the following special cases of (2.9). For $k = 1$, we have

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ c \otimes a^n & \text{if } a < 1. \end{cases} \tag{2.10}$$

In particular, for $c = 0$ we get globalization while for $c = 1$ and $n = 1$, we get identity. For $L = [0, 1]$ and $\otimes = \min$ (minimum t-norm), (2.10) yields $a^* = 1$ for $a = 1$, and for $a < 1$ we get $a^* = c$ if $a \geq c$ and $a^* = a$ if $a < c$.

To sum up, requiring “if it is (very) true that A is a subset of B then the closure of A is a subset of the closure of B ” to be true means requiring that the truth degree of “it is (very) true that A is a subset of B ” is less or equal to the truth degree of “the closure of A is a subset of the closure of B ”. Denoting the closure of A by $C(A)$, our monotony condition translates to $S(A, B)^* \leq S(C(A), C(B))$.

DEFINITION 2.2

An \mathbf{L}^* -closure operator (fuzzy closure operator with truth stresser $*$, or simply a fuzzy closure operator) on a set X is a mapping $C : L^X \rightarrow L^X$ satisfying

$$A \subseteq C(A), \tag{2.11}$$

$$S(A_1, A_2)^* \leq S(C(A_1), C(A_2)) \tag{2.12}$$

$$C(A) = C(C(A)) \tag{2.13}$$

for every $A, A_1, A_2 \in L^X$.

REMARK 2.3

(1) Since the only truth stresser $*$ on $L = \{0, 1\}$ is given by $0^* = 0$ and $1^* = 1$, one easily verifies that for $L = \{0, 1\}$, \mathbf{L}^* -closure operators coincide with ordinary closure operators.

(2) Note that if $*$ satisfies (2.4), then due to (2.2) and (2.5), our (2.12) is equivalent to $S(A, B)^* \leq S(C(A), C(B))^*$. Indeed, from $a^* \leq b$ we get $a^* = a^{**} \leq b^*$; conversely, if $a^* \leq b^*$ then $a^* \leq b$ due to $b^* \leq b$.

(3) Consider the boundary cases of $*$. For identity, (2.12) becomes $S(A, B) \leq S(C(A), C(B))$ which is the monotony condition used in [3] and [18] (but here, this condition was not formulated this way). For globalization, (2.12) says that if $A \subseteq B$, i.e. $S(A, B) = 1$, then $C(A) \subseteq C(B)$, i.e. $S(C(A), C(B)) = 1$. This is the commonly required condition which is

often the only required condition of monotony of fuzzy closure operators, see e.g. [13, 3] for an overview and references. With globalization, each \mathbf{L}^* -closure operator on X is a closure operator on the complete lattice $\langle L^X, \subseteq \rangle$.

(4) Clearly, if $*^1 \leq *^2$ (meaning that $a^{*^1} \leq a^{*^2}$ for each $a \in L$) then each \mathbf{L}^{*^2} -closure operator is an \mathbf{L}^{*^1} -closure operator. In particular, since globalization is the least truth stresser, properties of fuzzy closure operators with globalization can be considered common properties of fuzzy closure operators with truth stressers.

(5) We show that $*$ matters. Take \mathbf{L} where $L = [0, 1]$ with Gödel structure (i.e. \otimes is \min), $X = \{x_1, x_2\}$, and define C by $C(A)(x_1) = 0, C(A)(x_2) = 0.5$ for $A(x_1) = 0, A(x_2) \leq 0.5$, and $C(A)(x_1) = C(A)(x_2) = 1$ otherwise. An easy inspection shows that for $*$ being globalization, C is an \mathbf{L}^* -closure operator. However, if $*$ is the identity, C is not an \mathbf{L}^* -closure operator. Indeed, for A_1, A_2 given by $A_1(x_1) = A_2(x_1) = 0, A_1(x_2) = 1, A_2(x_2) = 0.5$ we have $S(A_1, A_2)^* = 0.5 > 0 = S(C(A_1), C(A_2))$.

Let us now consider two further examples of \mathbf{L}^* -closure operators.

EXAMPLE 2.4

(1) Let X and Y be sets of objects and attributes, I be an \mathbf{L} -relation between X and Y with $I(x, y)$ being the degree to which object x has attribute y . Introduce $\Delta : L^X \rightarrow L^Y$ and $\nabla : L^Y \rightarrow L^X$ by $A^\Delta(y) = \bigwedge_{x \in X} (A(x) \rightarrow I(x, y))$ and $B^\nabla(x) = \bigwedge_{y \in Y} (B(y)^* \rightarrow I(x, y))$, and put $\mathcal{B}(X, Y^*, I) = \{ \langle A, B \rangle \in L^X \times L^Y \mid A^\Delta = B, B^\nabla = A \}$. Then A^Δ is the fuzzy set of all attributes common to objects from A , B^∇ is the fuzzy set of all objects sharing all attributes for which it is very true that they belong to B . For $*$ being the identity, $\mathcal{B}(X, Y, I)$ is called a (fuzzy) concept lattice, pairs $\langle A, B \rangle \in \mathcal{B}(X, Y, I)$ are interpreted as concepts in the extent-intent (Port-Royal) approach [4]. For $*$ being globalization, $\mathcal{B}(X, Y^*, I) \subseteq \mathcal{B}(X, Y, I)$ and $\langle A, B \rangle \in \mathcal{B}(X, Y^*, I)$ are the “crisply generated” concepts [6]. $\mathcal{B}(X, Y^*, I)$ is studied in [6] where it is shown that $|\mathcal{B}(X, Y^*, I)|$ is usually much smaller than $|\mathcal{B}(X, Y, I)|$ which is an interesting fact since a concept lattice is a collection of all interesting clusters in data in the sense of formal concept analysis [12]. Now, the composed mapping $\Delta^\nabla : L^X \rightarrow L^X$ is an \mathbf{L}^* -closure operator whose fixed points are exactly fuzzy sets $A \in L^X$ such that $\langle A, B \rangle \in \mathcal{B}(X, Y^*, I)$ for some $B \in L^Y$.

(2) Another example of an \mathbf{L}^* -closure operator appears in fuzzy Horn logic [7]. Formulas in fuzzy Horn logic are of the form $\Delta((\bar{a}_1 \Rightarrow (t_1 \approx s_1)) \bar{\wedge} \cdots \bar{\wedge} (\bar{a}_n \Rightarrow (t_n \approx s_n))) \Rightarrow (t \approx s)$. Δ is a unary connective interpreted by truth stresser $*$ (with some additional properties), \bar{a}_i are truth constants interpreted by $a_i \in L$, $\bar{\wedge}$ and \Rightarrow are interpreted by \wedge and \rightarrow , s 's and t 's are terms interpreted as usual, and \approx is an \mathbf{L} -equality relation which is compatible with all functions. That is, structures (called \mathbf{L} -algebras) are sets equipped with functions and a binary \mathbf{L} -relation. The above formula can be written as $P \Rightarrow s \approx t$ with an \mathbf{L} -relation P given by $P(s_i, t_i) = a_i$. If Σ is an \mathbf{L} -theory of some class of \mathbf{L} -algebras, i.e. $\Sigma(P \Rightarrow s \approx t)$ is the degree to which $P \Rightarrow s \approx t$ is true in the class, then a mapping sending P to Σ_P , with $\Sigma_P(t, s) = \Sigma(P \Rightarrow s \approx t)$, is an \mathbf{L}^* -closure operator, see [7].

The following is an alternative definition of \mathbf{L}^* -closure operators.

THEOREM 2.5

$C : L^X \rightarrow L^X$ is an \mathbf{L}^* -closure operator on X iff it satisfies (2.11) and the following condition:

$$S(A_1, C(A_2))^* \leq S(C(A_1), C(A_2)). \tag{2.14}$$

PROOF. Assume (2.11)–(2.13). We get $S(A_1, C(A_2))^* \leq S(C(A_1), C(C(A_2))) = S(C(A_1), C(A_2))$, verifying (2.14). Conversely, assume (2.11) and (2.14). By (2.11), $A_2 \subseteq C(A_2)$, whence

$S(A_1, A_2) \leq S(A_1, C(A_2))$. Furthermore, (2.5) and (2.14) imply $S(A_1, A_2)^* \leq S(A_1, C(A_2))^* \leq S(C(A_1), C(A_2))$, proving (2.12). By (2.14), $1 = S(C(A), C(A))^* \leq S(C(C(A)), C(A))$, i.e. we have $C(C(A)) \subseteq C(A)$. Since the converse inclusion holds by (2.11), we conclude (2.13). ■

Note that in the proof of Theorem 2.5, only properties (2.1) and (2.5) were used.

DEFINITION 2.6

A system $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ is called *closed under S^* -intersections* iff for each $A \in L^X$ it holds that

$$\bigcap_{i \in I} (S(A, A_i)^* \rightarrow A_i) \in \mathcal{S}$$

where

$$\left(\bigcap_{i \in I} S(A, A_i)^* \rightarrow A_i\right)(x) = \bigwedge_{i \in I} (S(A, A_i)^* \rightarrow A_i(x))$$

for each $x \in X$. A system closed under S^* -intersections will be called an **L**^{*}-closure system.

REMARK 2.7

(1) Let $*$ be globalization, see (2.8). Then we have

$$\bigcap_{i \in I} S(A, A_i)^* \rightarrow A_i = \bigcap_{i \in I, A \subseteq A_i} A_i.$$

Therefore, \mathcal{S} is a **2**-closure system iff for each $A \subseteq X$ it holds $\bigcap_{A \subseteq A_i} A_i \in \mathcal{S}$. It can be easily seen that the last condition is equivalent to being closed under arbitrary intersections for \mathcal{S} . Hence, **2**-closure systems coincide with closure systems, i.e. systems of sets closed under S^* -intersections, where $*$ is given by above.

(2) In general, being closed under arbitrary intersections is a weaker condition than being closed under S^* -intersections. Indeed, let \mathcal{S} be closed under S^* -intersections. To show that \mathcal{S} is closed under arbitrary intersections, it suffices to show that

$$\bigwedge_{j \in J} A_j(x) = \bigwedge_{i \in I} (S(\bigcap_{j \in J} A_j, A_i)^* \rightarrow A_i(x))$$

holds for any $J \subseteq I$. The inequality \geq is clearly valid since for each $j' \in J$ we have $S(\bigcap_{j \in J} A_j, A_{j'})^* \rightarrow A_{j'}(x) = 1 \rightarrow A_{j'}(x) = A_{j'}(x)$. The converse inequality holds iff

$$\bigwedge_{j \in J} A_j(x) \leq S(\bigcap_{j \in J} A_j, A_i)^* \rightarrow A_i(x)$$

for each $i \in I$ which is equivalent to

$$\bigwedge_{j \in J} A_j(x) \otimes S(\bigcap_{j \in J} A_j, A_i)^* \leq A_i(x),$$

and because $S(\bigcap_{j \in J} A_j, A_i)^* \leq S(\bigcap_{j \in J} A_j, A_i)$ it suffices to show

$$\bigwedge_{j \in J} A_j(x) \otimes S(\bigcap_{j \in J} A_j, A_i) \leq A_i(x),$$

i.e.

$$\bigwedge_{j \in J} A_j(x) \otimes \left(\bigwedge_{y \in X} \left(\bigwedge_{j \in J} A_j(y) \rightarrow A_i(y) \right) \right) \leq A_i(x)$$

which is true because

$$\begin{aligned} & \bigwedge_{j \in J} A_j(x) \otimes \left(\bigwedge_{y \in X} \left(\bigwedge_{j \in J} A_j(y) \right) \rightarrow A_i(y) \right) \leq \\ & \leq \bigwedge_{j \in J} (A_j(x) \otimes \left(\left(\bigwedge_{j \in J} A_j(x) \right) \rightarrow A_i(x) \right)) \leq A_i(x). \end{aligned}$$

On the other hand, put $X = \{x, y\}$, take the Gödel structure with $\mathbf{L} = \{0, \frac{1}{2}, 1\}$, $a^* = a$, for any $a \in L$, $\mathcal{S} = \{\{0/x, 0/y\}, \{\frac{1}{2}/x, 0/y\}, \{1/x, 1/y\}\}$, and $A = \{1/x, 0/y\}$. Then \mathcal{S} is clearly closed under arbitrary intersections but not under S^* -intersections since $\bigcap S(A, A_i)^* \rightarrow A_i = A \notin \mathcal{S}$. Indeed, we have

$$S(A, \{ \frac{1}{2}/x, 0/y \} \rightarrow \{ \frac{1}{2}/x, 0/y \})^* = \frac{1}{2} \rightarrow \{ \frac{1}{2}/x, 0/y \} = \{ 1/x, 0/y \} \notin \mathcal{S}.$$

If $*$ satisfies (2.4), closedness under S^* -intersections is equivalent to closedness under intersections of “ a^* -shifted” \mathbf{L} -sets (notice that (2.1)–(2.3) do not matter for this to be true). Recall that for $a \in L$, $A \in L^X$, $a \rightarrow A$ denotes the \mathbf{L} -set defined by $(a \rightarrow A)(x) = a \rightarrow A(x)$.

THEOREM 2.8

If $*$ satisfies (2.4), then $\mathcal{S} \subseteq \mathcal{L}^X$ is an \mathbf{L}^* -closure system iff for any set I , $a_i \in L$, $A_i \in \mathcal{S}$ ($i \in I$), we have $\bigcap_{i \in I} (a_i^* \rightarrow A_i) \in \mathcal{S}$.

PROOF. If $\bigcap_{i \in I} (a_i^* \rightarrow A_i) \in \mathcal{S}$ for every $a_i \in L$, then taking $a_i = S(A, A_i)$ we get $\bigwedge_{i \in I} S(A, A_i)^* \rightarrow A_i(x) = \bigwedge_{i \in I} (a_i^* \rightarrow A_i(x))$, i.e. \mathcal{S} is an \mathbf{L}^* -closure system.

Let now \mathcal{S} be an \mathbf{L}^* -closure system. Take $a_i \in L$ and put $A = \bigcap_{i \in I} (a_i^* \rightarrow A_i)$. We have to show $A \in \mathcal{S}$. Clearly, it suffices to show that $\bigcap_{i \in I} (S(A, A_i)^* \rightarrow A_i) = A$. On the one hand, $\bigcap_{i \in I} (S(A, A_i)^* \rightarrow A_i) \supseteq A$ is true iff for each $i \in I$ we have $A \subseteq S(A, A_i)^* \rightarrow A_i$, i.e. iff for each $x \in X$, $S(A, A_i)^* \leq A(x) \rightarrow A_i(x)$ which is evident. The converse inclusion holds true iff for each $i \in I$ we have

$$\bigwedge_{i \in I} S(A, A_i)^* \rightarrow A_i(x) \leq a_i^* \rightarrow A_i(x),$$

i.e. iff

$$a_i^* \otimes \bigwedge_{i \in I} S(A, A_i)^* \rightarrow A_i(x) \leq A_i(x)$$

which is true since $a_i^* \otimes \bigwedge_{i \in I} S(A, A_i)^* \rightarrow A_i(x) \leq S(A, A_i)^* \otimes \bigwedge_{i \in I} S(A, A_i)^* \rightarrow A_i(x) \leq S(A, A_i)^* \otimes (S(A, A_i)^* \rightarrow A_i(x)) \leq A_i(x)$. The only step to verify here is to check that $a_i^* \leq S(A, A_i)^*$. Due to (2.4), it is enough to check $a_i^* \leq S(A, A_i)$ which is true iff for each $i \in I$ we have

$$a_i^* \leq \left(\bigwedge_{j \in I} (a_j^* \rightarrow A_j(x)) \right) \rightarrow A_i(x),$$

i.e. iff

$$a_i^* \otimes \left(\bigwedge_{j \in I} (a_j^* \rightarrow A_j(x)) \right) \leq A_i(x)$$

which is true. ■

COROLLARY 2.9

Provided $*$ satisfies (2.4), a system \mathcal{S} which is closed under arbitrary intersections is an \mathbf{L}^* -closure system iff for each $a \in L$ and $A \in \mathcal{S}$ it holds $a^* \rightarrow A \in \mathcal{S}$.

The following theorem shows that a closure of A in an \mathbf{L}^* -closure system can be obtained by intersections of supersets of A .

THEOREM 2.10

Let $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ be an \mathbf{L}^* -closure system. Then for each $A \in L^X$ it holds

$$\bigcap_{i \in I} S(A, A_i)^* \rightarrow A_i = \bigcap_{i \in I, A \subseteq A_i} A_i.$$

PROOF. As $S(A, A_i)^* = 1$ iff $S(A, A_i) = 1$ iff $A \subseteq A_i$, we have

$$\bigcap_{i \in I} S(A, A_i)^* \rightarrow A_i \subseteq \bigcap_{i \in I, S(A, A_i)^* = 1} S(A, A_i)^* \rightarrow A_i = \bigcap_{i \in I, A \subseteq A_i} A_i.$$

Conversely, since \mathcal{S} is an \mathbf{L}^* -closure system, we have $\bigcap_{i \in I} S(A, A_i)^* \rightarrow A_i = A_j \in \mathcal{S}$. Since $A \subseteq \bigcap_{i \in I} S(A, A_i)^* \rightarrow A_i$, we have $\bigcap_{i \in I, A \subseteq A_i} A_i \subseteq \bigcap_{i \in I} S(A, A_i)^* \rightarrow A_i$. \blacksquare

Note that we used only (2.5) in the proof.

LEMMA 2.11

Let $\mathcal{S} = \{A_i \mid i \in I\}$ be an \mathbf{L}^* -closure system. Then $C_{\mathcal{S}} : L^X \rightarrow L^X$ defined by

$$C_{\mathcal{S}}(A) = \bigcap_{i \in I} (S(A, A_i)^* \rightarrow A_i)$$

is an \mathbf{L}^* -closure operator. Moreover, for $A \in L^X$, we have $A \in \mathcal{S}$ iff $A = C_{\mathcal{S}}(A)$.

PROOF. We check (2.11)–(2.13).

(2.11): This was shown in the proof of Theorem 2.8.

(2.12): $S(A, B)^* \leq S(C_{\mathcal{S}}(A), C_{\mathcal{S}}(B))$ holds true iff for each $x \in X$ we have $S(A, B)^* \otimes C_{\mathcal{S}}(A)(x) \leq C_{\mathcal{S}}(B)(x)$ iff for each $i \in I$ we have

$$S(A, B)^* \otimes S(B, A_i)^* \otimes C_{\mathcal{S}}(A)(x) \leq A_i(x).$$

Now, since $S(A, B) \otimes S(B, A_i) \leq S(A, A_i)$, (2.5) and (2.6) give $S(A, B)^* \otimes S(B, A_i)^* \leq (S(A, B) \otimes S(B, A_i))^* \leq S(A, A_i)^*$. Therefore, $S(A, B)^* \otimes S(B, A_i)^* \otimes C_{\mathcal{S}}(A)(x) \leq S(A, A_i)^* \otimes (S(A, A_i)^* \rightarrow A_i(x)) \leq A_i(x)$, proving the required inequality.

(2.13): We have to show $C_{\mathcal{S}}(C_{\mathcal{S}}(A)) \subseteq C_{\mathcal{S}}(A)$. Since $C_{\mathcal{S}}(A) \in \mathcal{S}$, there is some $j \in I$ such that $A_j = C_{\mathcal{S}}(A)$. We therefore have

$$\begin{aligned} C_{\mathcal{S}}(C_{\mathcal{S}}(A))(x) &= \bigwedge_{i \in I} S(C_{\mathcal{S}}(A), A_i)^* \rightarrow A_i(x) \leq \\ &\leq S(C_{\mathcal{S}}(A), C_{\mathcal{S}}(A))^* \rightarrow (C_{\mathcal{S}}(A))(x) = 1 \rightarrow (C_{\mathcal{S}}(A))(x) = (C_{\mathcal{S}}(A))(x). \end{aligned}$$

We now show that $A \in \mathcal{S}$ iff $A = C_{\mathcal{S}}(A)$: If $A = A_j \in \mathcal{S}$ then $A_j \subseteq C_{\mathcal{S}}(A_j)$ as proved above. Conversely,

$$\begin{aligned} C_{\mathcal{S}}(A_j)(x) &= \bigwedge_{i \in I} (S(A_j, A_i)^* \rightarrow A_i(x)) \leq \\ &\leq S(A_j, A_j)^* \rightarrow A_j(x) = A_j(x), \end{aligned}$$

i.e. $C_{\mathcal{S}}(A_j) \subseteq A_j$. If $A = C_{\mathcal{S}}(A)$ then $A \in \mathcal{S}$ by the definition of an \mathbf{L}^* -closure system, completing the proof. \blacksquare

All of (2.1)–(2.3) are used in the proof of Lemma 2.11.

LEMMA 2.12

Let $C : L^X \rightarrow L^X$ be an \mathbf{L}^* -closure operator. Then $\mathcal{S}_C = \{A \in L^X \mid A = C(A)\}$ is an \mathbf{L}^* -closure system.

PROOF. Let I be such that $\mathcal{S}_C = \{A_i \mid i \in I\}$. We have to show that for each $A \in L^X$ we have $\bigcap_{i \in I} (S(A, A_i)^* \rightarrow A_i) \in \mathcal{S}_C$. To this end it suffices to show

$$\bigcap_{i \in I} (S(A, A_i)^* \rightarrow A_i) = C(A). \quad (2.15)$$

On the one hand, since $S(A, C(A)) = 1$, we have

$$\bigwedge_{i \in I} (S(A, A_i)^* \rightarrow A_i(x)) \leq S(A, C(A)) \rightarrow C(A)(x) = C(A)(x).$$

On the other hand,

$$C(A)(x) \leq \bigwedge_{i \in I} (S(A, A_i)^* \rightarrow A_i(x))$$

iff for each $i \in I$ it holds $C(A)(x) \otimes S(A, A_i)^* \leq A_i(x)$. This is true since

$$\begin{aligned} C(A)(x) \otimes S(A, A_i)^* &\leq C(A)(x) \otimes S(C(A), C(A_i)) \leq \\ &\leq C(A)(x) \otimes (C(A)(x) \rightarrow C(A_i)(x)) \leq C(A_i)(x) = A_i(x), \end{aligned}$$

completing the proof of (2.15). ■

In the proof of Lemma 2.12, we used only (2.1). The following theorem shows that there is a bijective correspondence between \mathbf{L}^* -closure operators and \mathbf{L}^* -closure systems on X .

THEOREM 2.13

Let C be an \mathbf{L}^* -closure operator on X , \mathcal{S} be an \mathbf{L}^* -closure system on X . Then \mathcal{S}_C is an \mathbf{L}^* -closure system on X , $C_{\mathcal{S}}$ is an \mathbf{L}^* -closure operator on X , and we have $C = C_{\mathcal{S}_C}$ and $\mathcal{S} = \mathcal{S}_{C_{\mathcal{S}}}$, i.e. the mappings $C \mapsto \mathcal{S}_C$ and $\mathcal{S} \mapsto C_{\mathcal{S}}$ are mutually inverse.

PROOF. By Lemmas 2.11 and 2.12 it remains to prove $C = C_{\mathcal{S}_C}$, i.e. that for any $A \in L^X$, $x \in X$,

$$C(A)(x) = \bigwedge_{A' \in L^X, A' = C(A')} (S(A, A')^* \rightarrow A'(x)).$$

Now, “ \leq ” holds iff for each $A' \in L^X$ such that $A' = C(A')$ we have $C(A)(x) \otimes S(A, A')^* \leq A'(x)$ which is true since $C(A)(x) \otimes S(A, A')^* \leq C(A)(x) \otimes S(C(A), C(A')) \leq C(A')(x) = A'(x)$. Conversely, putting $A' = C(A)$ we get

$$S(A, C(A))^* \rightarrow C(A)(x) = 1 \rightarrow C(A)(x) = C(A)(x),$$

proving “ \geq ”. ■

3 Fuzzy interior operators and other closure structures

The monotony condition “if a fuzzy set A is (somehow) included in B then $f(A)$ is (somehow) included in $f(B)$ ” (with f being some operator) and its variations appear in various situations in fuzzy sets and their applications. For instance, the so-called extensional operators, i.e. operators f satisfying $E(A, B) \leq E(f(A), f(B))$, are of interest in approximate reasoning. If $f(A)$ is the (output) fuzzy set obtained by some processing (inference) from the (input) fuzzy set A , then $E(A, B) \leq E(f(A), f(B))$ says that similar inputs lead to similar outputs. In these situations, using truth stressers in the same way as we do here seems to be a reasonable way to have a parameterized monotony. This is open for investigation. In this section, we show that the $*$ -monotony works for interior operators on fuzzy sets as well.

In the bivalent case, closure operators are in a one-to-one relationship with so-called interior operators. Recall that an interior operator on X is a mapping $I : \mathbf{2}^X \rightarrow \mathbf{2}^X$ satisfying (1) $I(A) \subseteq A$; (2) $A \subseteq B$ implies $I(A) \subseteq I(B)$; (3) $I(A) = I(I(A))$ for any subsets A and B of X . It is a well known fact that given an interior operator I and a closure operator C , putting $C_I(A) = I(\overline{A})$ and $I_C(A) = C(\overline{A})$, C_I is a closure operator and I_C is an interior operator (here, \overline{A} denotes the complement of A). This fact is a consequence of the law of double negation (i.e. $a = (a \rightarrow 0) \rightarrow 0$) which is true in $\mathbf{2}$ (structure of bivalent logic) but is not true in a general \mathbf{L} (structures of fuzzy logic). Therefore, in a fuzzy setting, we cannot obtain results on interior operators automatically from results on closure operators and what is needed is to develop the theory of interior operators from scratch.

Fuzzy interior operators appear in several studies, e.g. [2, 5, 10]. In the context of Section 2, we are interested in the following notion.

DEFINITION 3.1

An \mathbf{L}^* -interior operator on a set X is a mapping $I : L^X \rightarrow L^X$ satisfying

$$I(A) \subseteq A, \quad (3.1)$$

$$S(A_1, A_2)^* \leq S(I(A_1), I(A_2)) \quad (3.2)$$

$$I(A) = I(I(A)) \quad (3.3)$$

for every $A, A_1, A_2 \in L^X$.

In what follows, we are going to show that the main results of Section 2 have their analogies for interior operators. However, since the ideas of the proofs are similar to those from Section 2, we list only the main results with comments. We again assume that $*$ satisfies (2.1)–(2.3).

THEOREM 3.2

$I : L^X \rightarrow L^X$ is an \mathbf{L}^* -interior operator on X iff it satisfies (3.1) and the following condition:

$$S(I(A_1), A_2)^* \leq S(I(A_1), I(A_2)). \quad (3.4)$$

DEFINITION 3.3

A system $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ is called *closed under S^* -unions* iff for each $A \in L^X$ we have

$$\bigcup_{i \in I} S(A, A_i)^* \otimes A_i \in \mathcal{S}$$

where

$$\left(\bigcup_{i \in I} S(A, A_i)^* \otimes A_i \right)(x) = \bigvee_{i \in I} (S(A, A_i)^* \otimes A_i(x))$$

for each $x \in X$. A system closed under S^* -unions will be called an L^* -interior system.

Assuming (2.4), closedness under S^* -unions is equivalent to closedness under unions of “ a^* -cut” L -sets. Recall that for $a \in L$, $A \in L^X$, $a \otimes A$ is defined by $(a \otimes A)(x) = a \otimes A(x)$.

THEOREM 3.4

Provided $*$ satisfies (2.4), a system \mathcal{S} which is closed under arbitrary unions is an L^* -interior system iff for each $a \in L$ and $A \in \mathcal{S}$ it holds $a^* \otimes A \in \mathcal{S}$.

Finally, we have a one-to-one relationship between L^* -interior operators and L^* -interior systems.

THEOREM 3.5

Let I be an L^* -interior operator on X , \mathcal{S} be an L^* -interior system on X . Then $\mathcal{S}_I = \{A \in L^X \mid A = I(A)\}$ is an L^* -interior system on X , $I_{\mathcal{S}}$ defined by $I_{\mathcal{S}}(A) = \bigvee_{i \in I} S(A_i, A)^* \otimes A_i(x)$ is an L^* -interior operator on X , and we have $I = I_{\mathcal{S}_I}$ and $\mathcal{S} = \mathcal{S}_{I_{\mathcal{S}}}$, i.e. the mappings $I \mapsto \mathcal{S}_I$ and $\mathcal{S} \mapsto I_{\mathcal{S}}$ are mutually inverse.

Acknowledgement Supported by grant no. B1137301 of GA AV ČR and by institutional support, research plan MSM 6198959214.

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