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An answer to Demirci's open question, a clarification of his result, and a correction of his interpretation of the result

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Abstract

In his note (this issue of this journal), Demirci shows that fields with fuzzy equalities have only trivial fuzzy equalities, concludes that "Therefore, in case **L**-algebras contain field structure, all results in [R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic/Plenum Publishers, New York, 2002; R. Bělohlávek, V. Vychodil, Algebras with fuzzy equalities, in: proceedings of the 10th IFSA world congress, June 29–July 2, 2003, pp. 1–4; R. Bělohlávek, V. Vychodil, Algebras with fuzzy equalities, Fuzzy Sets and Systems, accepted; V. Vychodil, Direct limits and reduced products of algebras with fuzzy eqalities, submitted] are evident from their classical counterparts", and asks a question "does there exist any **L**-algebra with an **L**-equality different from trivial **L**-equalities in case the ordinary part of the **L**-algebra includes two binary operations that define group, ring, module or vector space structure?" In our reply, we show the following. First, by presenting examples of group-based **L**-algebras with non-trivial **L**-equalities, we show that the answer to Demirci's question is positive. Second, we clarify the meaning of Demirci's result and show that it is in fact a natural generalization of the well-known classical result saying that ordinary fields do not have non-trivial congruences. Third, we argue that Demirci's interpretation of his result is mistaken and that it is not true that "all results in [R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic/Plenum Publishers, New York, 2002; R. Bělohlávek, V. Vychodil, Algebras with fuzzy equalities, Fuzzy Sets and Systems, to appear; R. Bělohlávek, V. Vychodil, Fuzzy Equational Logic, Springer, Berlin, to appear; S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer, New York, 1981] are evident from their classical counterparts."

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1. Answer to Demirci's open question

Summary: The answer to Demirci's question is yes.

Details:

Demirci's question: "does there exist any **L**-algebra with an **L**-equality different from trivial **L**-equalities in case the ordinary part of the **L**-algebra includes two binary operations that define group, ring, module or vector space structure?" (quote from abstract of [5])

Although we do not agree with and somewhat do not understand Demirci saying that "This question is equivalent to the problem of whether the notion of an **L**-algebra provides a meaningful generalization of ring, module or vector space structure in the context of fuzzy equality", we agree that it is important to know whether there are such examples. Here are two examples of **L**-algebras with group structure, the first one a group with fuzzy equality, the second one a ring with fuzzy equality.

Example 1 (*Permutation groups with fuzzy equality and a particular example*). Let U be a set equipped with an **L**-equality $\approx^U$. Let $M = S(U)$ be the set of all permutations of U compatible with $\approx^U$, i.e. the set of all bijective mappings π on U for which we have $(u \approx^U v) \leqslant (\pi(u) \approx^U \pi(v))$. The triplet $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, \circ^{\mathbf{M}} \rangle$ where

$$\pi \approx^{\mathbf{M}} \sigma = \bigwedge_{u,v} \big((u \approx^U v) \to (\pi(u) \approx^U \sigma(v)) \big) \tag{1}$$

and $\circ^{\mathbf{M}}$ denotes the composition of permutations is an **L**-algebra. Note that $\pi \approx^{\mathbf{M}} \sigma$ is the degree to which it is true that similar elements are mapped to similar ones. Moreover, the restriction of **M** to mappings satisfying $(u \approx^U v) = (\pi(u) \approx^U \pi(v))$ is a group with fuzzy equality. This was proved in [2].

To see a particular example, let **L** be the standard Łukasiewicz algebra on the unit interval. Consider $U = \{a, \ldots, f\}$, and let $\approx^U$ be given by Fig. 1. There are the four compatible permutations on U:

$$\pi_1 = \mathrm{id}_U, \quad \pi_2 = \begin{pmatrix} a & b & c & d & e & f \\ a & b & c & f & e & d \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} a & b & c & d & e & f \\ b & a & c & d & e & f \end{pmatrix},$$
$$\pi_4 = \begin{pmatrix} a & b & c & d & e & f \\ b & a & c & f & e & d \end{pmatrix}.$$

The resulting **L**-algebra $\mathbf{M} = \langle \{\pi_1, \ldots, \pi_4\}, \approx^{\mathbf{M}}, \circ^{\mathbf{M}} \rangle$ of compatible permutations is a group (so-called Klein group) with fuzzy equality and is depicted in Fig. 1.

Remark 2. In the abstract, Demirci asks for **L**-algebras where the ordinary part "includes two binary operations that define group, ring, …". We are not sure how to understand this since groups are usually considered as algebras with one binary operation, or one binary operation, one unary operation (operation of an inverse element), and possibly one nullary operation (neutral element). One can add a unary operation of an inverse element and a nullary operation of a neutral element in Example 1. An inverse element of each π_i is π_i itself, a neutral element is π_1.

$\approx^U$	a	b	c	d	e	f
a	1	3/4	3/4	0	1/4	0
b	3/4	1	3/4	0	1/4	0
c	3/4	3/4	1	1/4	1/2	1/4
d	0	0	1/4	1	3/4	7/8
e	1/4	1/4	1/2	3/4	1	3/4
f	0	0	1/4	7/8	3/4	1

$\circ^{\mathbf{M}}$	π_1	π_2	π_3	π_4
π_1	π_1	π_2	π_3	π_4
π_2	π_2	π_1	π_4	π_3
π_3	π_3	π_4	π_1	π_2
π_4	π_4	π_3	π_2	π_1

$\approx^{\mathbf{M}}$	π_1	π_2	π_3	π_4
π_1	1	7/8	3/4	3/4
π_2	7/8	1	3/4	3/4
π_3	3/4	3/4	1	7/8
π_4	3/4	3/4	7/8	1

Fig. 1. Group with fuzzy equality.

$\approx^{\mathbf{M}}$	a	b	c	d
a	1.0	0.8	0.5	0.5
b	0.8	1.0	0.5	0.5
c	0.5	0.5	1.0	0.8
d	0.5	0.5	0.8	1.0

$+^{\mathbf{M}}$	a	b	c	d
a	c	d	b	a
b	d	c	a	b
c	b	a	d	c
d	a	b	c	d

	$-^{\mathbf{M}}$
a	b
b	a
c	c
d	d

$\cdot^{\mathbf{M}}$	a	b	c	d
a	b	a	c	d
b	a	b	c	d
c	c	c	d	d
d	d	d	d	d

Fig. 2. Ring with fuzzy equality.

Example 3 (*Ring with fuzzy equality*). Fig. 2 shows an example of a ring (commutative ring with a unit) $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, +^{\mathbf{M}}, -^{\mathbf{M}}, \cdot^{\mathbf{M}}\rangle$ with fuzzy equality with $M = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$. $+^{\mathbf{M}}$ is the additive group operation, d is the ring zero (neutral element w.r.t. $+^{\mathbf{M}}$), $-^{\mathbf{M}}$ is the unary operation of an inverse element (w.r.t. $+^{\mathbf{M}}$), $\cdot^{\mathbf{M}}$ is ring multiplication. **M** has a unit element b (neutral element w.r.t. $\cdot^{\mathbf{M}}$).

2. Clarification of Demirci's result

Summary: A reformulation of Demirci's result is the following: fields with fuzzy equalities have only trivial congruences. This generalizes a well-known fact that ordinary fields have only trivial congruences. Therefore, the content of [5] is the following: Demirci has found an example of **L**-algebras with special congruence properties and asks whether there are **L**-algebras which do not have these properties. Section 1 showed that the answer is yes.

Details: Recall that Demirci's result [5, Theorem 1] says the following.

Theorem 4. *If* $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ *is an* **L**-*algebra such that for some binary operations* $\circ^{\mathbf{M}}, \Delta^{\mathbf{M}} \in F^{\mathbf{M}}$, *the ordinary algebra* $\langle M, \circ^{\mathbf{M}}, \Delta^{\mathbf{M}}\rangle$ *is a field, then there is* $a \in L - \{1\}$ *such that* $\approx^{\mathbf{M}}$ *equals* $\approx^{\mathbf{M}}_a$.

Here, for $a \in L$, $\approx^{\mathbf{M}}_a$ is an **L**-equivalence in *M* defined by $u \approx^{\mathbf{M}}_a v = 1$ for $u = v$ and $u \approx^{\mathbf{M}}_a v = a$ for $u \neq v$. It is immediate that for $a = 1$, $\approx^{\mathbf{M}}_a$ is the largest **L**-equivalence in *M* and this is not an **L**-equality. For $a < 1$, $\approx^{\mathbf{M}}_a$ is an **L**-equality in *M*. In particular, $\approx^{\mathbf{M}}_0$ can be identified with a crisp identity id_M in *M*.

In order to look closely at the meaning of Theorem 4, consider the following well-known classical result [6].

Theorem 5 (*Classical result about fields*). *Each field* $\mathbf{M} = \langle M, F^{\mathbf{M}}\rangle$ *has only trivial congruences.*

That is, each congruence on a field $\mathbf{M} = \langle M, F \rangle$ is a trivial congruence, i.e. either it is id_M or $M \times M$. id_M and $M \times M$ are called trivial since they are the only equivalence relations which are congruences on any algebra with support set M. In accordance with the ordinary case, let us call a congruence on an $\mathbf{L}$-algebra $\mathbf{M} = \left\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \right\rangle$ trivial if it equals $\approx_a^{\mathbf{M}}$ for some $a \in L$. The motivation for this is clear. Namely, $\approx_a^{\mathbf{M}}$ is a congruence on any $\mathbf{L}$-algebra $\mathbf{M}$ with a support set M and an $\mathbf{L}$-equality $\approx^{\mathbf{M}}$ with $\approx^{\mathbf{M}} \subseteq \approx_a^{\mathbf{M}}$. Call a field with fuzzy equality any $\mathbf{L}$-algebra $\mathbf{M}$ such that its ordinary part (functional part) $\left\langle M, F^{\mathbf{M}} \right\rangle$ is an ordinary field. Then, it is easy to see that Theorem 4 is equivalent to the following claim.

Theorem 6 (*Reformulation of Demirci's result*). *Each field* $\mathbf{M} = \left\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \right\rangle$ *with fuzzy equality has only trivial congruences.*

Namely, $\mathbf{M} = \left\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \right\rangle$ is an $\mathbf{L}$-algebra if an only if $\approx^{\mathbf{M}}$ is a congruence on the $\mathbf{L}$-algebra $\mathbf{M} = \left\langle M, \approx_0^{\mathbf{M}}, F^{\mathbf{M}} \right\rangle$. Put in this perspective, Demirci's Question says the following:

Reformulation of Demirci's question: "We have examples of $\mathbf{L}$-algebras (fields with fuzzy equalities) which have only trivial congruences. Do there exist any $\mathbf{L}$-algebras with non-trivial congruences at all?"

We have seen in Section 1 that they do. Note that neither Demirci's result, nor the fact that the answer to his question is positive are surprising. Namely, Demirci's result yields a natural generalization of a well-known classical theorem about fields. The position of this result in the theory of algebras with fuzzy equalities should be considered the same way as the position of the classical result (Theorem 5) in the theory of ordinary algebras. The result is a particular example of the following fact which is quite expected.

Observation 7. *For several particular examples of* $\mathbf{L}$-*algebras, the special properties of operations of these* $\mathbf{L}$-*algebras influence properties of their congruences.*

Another example is the following assertion.

Lemma 8. *If* $\mathbf{M}$ *is a group with fuzzy equality* $\approx^{\mathbf{M}}$ *with operations* $\circ$ *and* $^{-1}$ *then for each* $\mathtt{a}, \mathtt{b} \in M$ *we have*

$$\mathtt{a} \approx^{\mathbf{M}} \mathtt{b} = \mathtt{a}^{-1} \approx^{\mathbf{M}} \mathtt{b}^{-1}.$$

Proof. We have $\mathtt{a} \approx^{\mathbf{M}} \mathtt{b} \leqslant \mathtt{a}^{-1} \approx^{\mathbf{M}} \mathtt{b}^{-1} \leqslant (\mathtt{a}^{-1})^{-1} \approx^{\mathbf{M}} (\mathtt{b}^{-1})^{-1} = \mathtt{a} \approx^{\mathbf{M}} \mathtt{b}$. □

No doubt there are other results which may be seen as concrete examples of Observation 7.

3. Correction of Demirci's interpretation of his result

Summary: Demirci's conclusions based on his result are exaggerated and mistaken.

Details: Based on his result, Demirci concludes that

Claim A. *"any* **L***-algebra* $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ *with the ordinary algebra* $\langle M, F^{\mathbf{M}} \rangle$ *including the field structure corresponds to its ordinary part* $\langle M, F^{\mathbf{M}} \rangle$ *in a one-to-one manner."*

And then

Claim B. *"Therefore, in case* **L***-algebras contain field structure, all results in* [1–4] *are evident from their classical counterparts."*

We argue that this interpretation of Theorem 4 is grossly incorrect and that the conclusion is mistaken.

Let us first note that it is not quite clear what Claim A means. Claim A suggests that for a given ordinary algebra $\langle M, F^{\mathbf{M}} \rangle$ including the field structure there is exactly one **L**-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$. However, this is not true since in this case, $\mathbf{M} = \langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ is an **L**-algebra for each $a \in L - \{1\}$ by Demirci's result (Theorem 4). Therefore, for a given $\langle M, F^{\mathbf{M}} \rangle$, there exists a whole collection of **L**-algebras $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$, namely, $\{\mathbf{M} = \langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle \mid a \in L - \{1\}\}$. In addition to that, for $a \neq b$, $\langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ is not isomorphic to $\langle M, \approx_b^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ (see [2] for a definition of isomorphism of **L**-algebras). The fact that $\langle M, F^{\mathbf{M}} \rangle$ and $\langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ seem to be almost identical might suggest that for fields with fuzzy equalities, "everything reduces to the ordinary case". However, one has to be careful and say exactly which properties of fields with fuzzy equalities follow trivially from the properties of ordinary fields. Namely, in general, algebraic properties of $\langle M, F^{\mathbf{M}} \rangle$ are different from those of $\mathbf{M} = \langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle$'s. For instance, the set $\mathrm{Con}(\langle M, F^{\mathbf{M}} \rangle)$ of all congruences on $\langle M, F^{\mathbf{M}} \rangle$ has just two congruences, namely id_M and $M \times M$. Contrary to that, the set $\mathrm{Con}(\mathbf{M})$ of all congruences on $\mathbf{M} = \langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ can be infinite. Namely, $\approx_b^{\mathbf{M}}$ is a congruence on $\mathbf{M} = \langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ for each $b \geqslant a$. Therefore (cf. also Section 2), for $\mathbf{M} = \langle M, \approx_0^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ we have $\mathrm{Con}(\mathbf{M}) = \{\approx_b^{\mathbf{M}} \mid b \in L\}$, and so $|\mathrm{Con}(\mathbf{M})| = |L|$. It also follows that $|\mathrm{Con}(\mathbf{M})| = |\mathrm{Con}(\langle M, F^{\mathbf{M}} \rangle)| = 2$ if and only if **L** is a two-element Boolean algebra (ordinary case). Once again, unless one clearly describes how the properties of a field $\mathbf{M} = \langle M, \approx_a^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ with fuzzy equality are related to properties of the ordinary field $\langle M, F^{\mathbf{M}} \rangle$, one needs to be careful with conclusions based on Theorem 4.

Demirci makes only one conclusion based on Theorem 4, namely Claim B. He does not supply any justification of Claim B except that it follows from Theorem 4. This suggests that Claim B follows from Theorem 4 in some obvious way. Now, we argue that *Claim B is false*. Although Claim A, which Demirci uses as an intermediate step between Theorem 4 and Claim B, is not clear, Claim B is a statement specific enough to argue that it is false. To be clear, we argue that it is not true that Demirci's result (Theorem 4) implies that all results of [2] (which is but one of the four papers to which Demirci refers by [1–4]) are evident from their classical counterparts.

Remark 9. The matter is somewhat delicate since we want to show that it is not the case that a statement is evident from another statement and "being evident" is subjective. Even if a reader is not convinced by the following arguments, the fact remains true that Demirci did not provide any justification for his Claim B and unless he does so, Claim B cannot be taken for true.

Consider [2, Theorem 3.1] which says that $\langle \mathrm{Con}(\mathbf{M}), \subseteq \rangle$ (the lattice of all congruences of an **L**-algebra **M**) is a complete sublattice of $\langle \mathrm{Eq}(M), \subseteq \rangle$ (the lattice of all **L**-equivalence relations on M). This assertion is proved for general **L**-algebras in [2] and it generalizes a useful result known for ordinary algebras [4].

Note also that for an $\mathbf{L}$-algebra $\mathbf{M}$ which is a field with fuzzy equality, this result can be proved more easily using Theorem 6 (details omitted). However, the point is that it is not the case that the assertion "for a field $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ with fuzzy equality, $\langle \mathrm{Con}(\mathbf{M}), \subseteq \rangle$ is a complete sublattice of $\langle \mathrm{Eq}(M), \subseteq \rangle$" is evident from the assertion "for each ordinary filed $\mathbf{N} = \langle N, F^{\mathbf{N}} \rangle$, $\langle \mathrm{Con}(\mathbf{N}), \subseteq \rangle$ (the lattice of ordinary congruences of $\mathbf{N}$) is a complete sublattice of $\langle \mathrm{Eq}(N), \subseteq \rangle$ (the lattice of ordinary equivalences on N)". Theorem 4 implies that $\approx^{\mathbf{M}}$ equals $\approx_a^{\mathbf{M}}$ for some $a \in L - \{1\}$. Now, in order to get the desired result by a straightforward application of its classical counterpart one could wish to take an appropriate ordinary field $\mathbf{N}$ such that $\langle \mathrm{Con}(\mathbf{N}), \subseteq \rangle$ is isomorphic to $\langle \mathrm{Con}(\mathbf{M}), \subseteq \rangle$ and $\langle \mathrm{Eq}(N), \subseteq \rangle$ is isomorphic to $\langle \mathrm{Eq}(M), \subseteq \rangle$. Theorem 4 suggests to take $\mathbf{N} := \langle M, F^{\mathbf{M}} \rangle$. However, this does not help since as we have seen, it may happen that $|\mathrm{Con}(\mathbf{M})| > |\mathrm{Con}(\mathbf{N})| = 2$. Therefore, it is by far not clear why Claim B should be true and the above discussion brings arguments for Claim B being false.

Remark 10. To make our point clear: our results to which Demirci refers show properties of general $\mathbf{L}$-algebras. It is, of course, true for both ordinary algebras as well as for $\mathbf{L}$-algebras that for special algebras (like fields, groups, lattices), some properties true for general algebras get simplified for the special algebras (the assertions or their proofs get more simple). For instance, there are algebras $\mathbf{M}$ which have no non-trivial subuniverses. For such algebras, the claim saying that the set $\mathrm{Sub}(\mathbf{M})$ of all subuniverses of a given algebra is a complete lattice becomes almost trivial. Properties get simplified also in case of fields with fuzzy equalities. For instance, the fact that the set $\mathrm{Con}(\mathbf{M})$ of all congruences of a field $\mathbf{M}$ with fuzzy equality $\approx_0^{\mathbf{M}}$ is a complete lattice follows easily from the above observation that $\langle \mathrm{Con}(\mathbf{M}), \subseteq \rangle$ is isomorphic to $\langle L, \leqslant \rangle$ and the fact that $\langle L, \leqslant \rangle$ is a complete lattice. However, this is not what Demirci's Claim B says. Claim B says that because of Theorem 4, all general algebraic results (from the cited papers) in their setting for fields with fuzzy equalities follow from their classical counterparts.

4. Further remarks

We conclude by the following remarks.

- The title of Demirci's paper is misleading. Demirci shows a non-existence of non-trivial congruences and not a non-existence of non-trivial $\mathbf{L}$-algebras as suggested by the title of [5]. Namely, following the ordinary case, an $\mathbf{L}$-algebra is called trivial [2] if its support set is a singleton. Therefore, the concept of a triviality of $\mathbf{L}$-algebras is set by definition in [2] which comes from an established concept in the ordinary case [4, p. 23, term "trivial algebra" included in subject index]. As such, it should not be redefined as Demirci does in the title of his paper.
- Note that the generic part of Example 1 describing a construction of a group with fuzzy equality (not the particular example we presented in this paper) was present in our paper [2] to which Demirci refers.
- Demirci puts the answer to his question as a test of whether the concept of an "$\mathbf{L}$-algebra is a meaningful generalization for what kinds of algebraic structures". Our examples showed that the answer to the question is positive. Note also that the examples of groups with fuzzy equalities (permutation groups of a set with fuzzy equality) and examples of lattices with fuzzy equalities (fixed points of fuzzy closure operators) are not artificially constructed and come from well-known structures in fuzzy set theory. In addition to the examples, we would like to turn the attention of an interested reader to [3] where one can find further examples including their graphical illustration, development of most of

model-theoretic properties that have been worked out for ordinary algebras in their setting for **L**-algebras, and logical calculi with their completeness results for reasoning with identities in the presence of fuzziness.

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References

[1] R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic/Plenum Publishers, New York, 2002.
[2] R. Bělohlávek, V. Vychodil, Algebras with fuzzy equalities, Fuzzy Sets and Systems, to appear, doi:10.1016/j.fss.2005.05.044.
[3] R. Bělohlávek, V. Vychodil, Fuzzy Equational Logic, Springer, Berlin, 2005, to appear.
[4] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer, New York, 1981.
[5] M. Demirci, Non-existence of non-trivial *L*-algebras containing field structure in the sense of Bělohlávek and an open question, Fuzzy Sets and Systems, to appear, doi:10.1016/j.fss.2005.06.010.
[6] J. Lambek, Lectures on Rings and Modules, Blaisdell, Waltham, MA, 1966.