On Boolean factor analysis with formal concepts as factors

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***Abstract*— The paper presents results on factorization of binary matrices using formal concepts generated from the input matrix. We prove several results, e.g. that each matrix is concept-factorizable, that concept-factorizability is the best way to factorize binary matrices, describe sets of good factors, mandatory factors, etc. In addition to that, we outline an algorithm for finding factors. The results are illustrated in detail by examples.**

***Keywords*— factor analysis, Boolean attribute, concept lattice, formal concept**

I. Introduction and Problem Setting

Factor analysis originated with Spearman's monumental development of psychological theory involving a single general factor and a number of specific factors [16]. Today, factor analysis is a well-established branch of statistical data analysis with many applications in numerous fields and with support in several software packages, see e.g. [1], [10]. According to Harman [10, p. 4], "The principal concern of factor analysis is the resolution of a set of variables linearly in terms of (usually) a small number of categories or 'factors'. … A satisfactory solution will yield factors which convey all the essential information of the original set of variables. Thus, the chief aim is to attain scientific parsimony or economy of description."

For our purpose, consider the following (somewhat simplified) formulation of the problem of a factor analysis. Suppose Z is an $n \times m$ real-valued matrix with entries $Z_{ij} \in \mathbb{R}$ ($1 \leq i \leq n$, $1 \leq j \leq m$) describing a value of j-th variable on i-th object. The aim is to find a collection of other variables $f_1, \dots, f_k$, so called factors, such that the original variables can be approximately expressed by a linear combination of the factors. In more detail, the aim is to find an $n \times k$ matrix A and a $k \times m$ matrix F such that

$$Z \approx A \circ F,$$

i.e. Z is approximately equal to $A \circ F$ ($\circ$ is the usual matrix product). Each A's entry $A_{il} \in \mathbb{R}$, called a loading, represents value of l-th factor on i-th object; each F's entry $F_{lj} \in \mathbb{R}$ represents how j-th variable is manifested in l-th factor. An important feature of the classical model of factor analysis is its linearity: Z is (approximately equal to) the ordinary matrix product of A and F, i.e.

$$Z_{ij} \approx \sum_{l=1}^{k} A_{il} \cdot F_{lj},$$

i.e. each original variable is considered as a linear combination of new variables (factors). Moreover, one requires that the number of factors is smaller than the number of the original variables, i.e. $k < m$. Then, the collection of all factors can be considered as a collection of a relatively small number of variables representing latent features such that the directly observable features can be expressed in terms of the latent features. Making the latent features explicit is advantageous. First, the dimension of the model reduces (the objects are represented in a k-dimensional space rather than in an m-dimensional one). Second, the factors can be thought of as representing more fundamental (explanatory) features than those represented by the original variables.

Needless to say, the general idea of factor analysis, i.e. finding a relatively small set of latent features, is independent of the linearity of the model. That is to say, the original variables can be expressed by means of the factors also in other ways than by a linear combination. In fact, there have been proposed several other models of factor analysis, in addition to the basic linear model. In this paper, we are interested in a so-called Boolean factor analysis (BFA), see e.g. [11], [15], [17]. The input data to Boolean factor analysis is an $n \times m$ matrix I describing a relationship between n objects and m binary attributes. Each entry I_{ij} ($1 \leq i \leq n$, $1 \leq j \leq m$) is a value 0 or 1. Value 1 of I_{ij} indicates that the i-th object has the j-th attribute, value 0 indicates that the i-th object does not have the j-th attribute. The aim is to find an $n \times k$ binary matrix A (i.e., $A_{ik} \in \{0, 1\}$) and a $k \times m$ binary matrix B (i.e., $B_{kj} \in \{0, 1\}$) such that

$$I \approx A \circ B,$$

where $\approx$ denotes "approximately equal" and $\circ$ denotes a Boolean matrix multiplication. That is,

$$I_{ij} = \bigvee_{l=1}^{k} A_{ik} \otimes B_{kj} \qquad (1)$$

where $\otimes$ denotes a logical "and" (truth function of conjunction, i.e., $1 \otimes 1 = 1$, $1 \otimes 0 = 0 \otimes 1 = 0 \otimes 0 = 0$) and $\bigvee$ denotes logical "or" (truth function of disjunction, i.e., $0 \otimes 0 = 0$, $1 \otimes 0 = 0 \otimes 1 = 1 \otimes 1 = 1$). For instance, we have

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here again, in addition to the n original objects and their m original binary attributes, we have k new factors. Matrices A and B represent relationships between objects and factors and between factors and the original attributes, respectively. In particular, $A_{il} = 1$ ($A_{il} = 0$) means that the i-th object has (does not have) the l-th factor, i.e. that the l-th factor is (is not) manifested on the i-th object; $B_{lj} = 1$ ($B_{lj} = 0$) means that the l-th factor subsumes (does not subsume) the j-th original attribute, i.e. that the j-th variable is (is not) manifested on the l-th factor. Therefore, instead of the classical factor analysis, Boolean factor analysis is suitable for presence/absence data. The factors, however, can again be seen as representing latent features using which one can express both the original objects and the original (directly observable) attributes.

There have been several approaches to Boolean factor analysis presented in the literature. In a series of papers, see e.g. [15], [17], the authors made use of a Hopfield-like neural networks for the purpose of Boolean factor analysis with factors being some of the attractors of a Hopfield-like neural network associated with the original matrix I. In the present paper, we are interested in an approach proposed in [11]. The approach consists in considering a so-called concept lattice associated to the input matrix I and in selecting some of its elements, so-called formal concepts, as the factors. This approach is quite appealing since formal concepts can be naturally interpreted as human-understandable concepts. Therefore, having factors as formal concepts is transparent and "user-friendly". The authors in [11] experimentally demonstrated that Boolean factorization using formal concepts yields promising results. However, no theoretical insight was provided in [11] either in order to justify the usage of formal concepts as factors or in order to speed up the search for suitable factors. The main aim of this paper is to provide a theoretical analysis of Boolean factor analysis which uses formal concepts as factors. We concentrate on exact factorization, i.e. on the case when we require $I = A \circ B$. As it will become apparent, however, our results are important also for the general case $I \approx A \circ B$.

In Section II, we provide a survey on formal concept analysis (formal concepts, concept lattices, etc.). In Section III-A, we present basic results. Section III-B describes an algorithm for construction of factors. Section III-C contains illustrative examples. In Section IV, we present conclusions, remarks, and an outline of future research.

II. Preliminaries

In this section, we summarize basic notions of formal concept analysis (FCA). We refer the reader to [8] and also to [6] for further details. FCA is a method of exploratory data analysis. The input data consists of a data table describing a relationship between objects and attributes. The output of FCA consists of a hierarchically ordered collection of clusters. The clusters are called formal concepts and can be seen as natural concepts well-understandable and interpretable by humans. The details follow.

An object-attribute data table describing which objects have which attributes can be identified with a triplet $\langle X, Y, I\rangle$ where X is a non-empty set (of objects), Y is a non-empty set (of attributes), and $I \subseteq X \times Y$ is an (object-attribute) relation. Objects and attributes correspond to table rows and columns, respectively, and $\langle x, y\rangle \in I$ indicates that object x has attribute y (table entry corresponding to row x and column y contains 1; if $\langle x, y\rangle \notin I$ the table entry contains 0). In the terminology of FCA, a triplet $\langle X, Y, I\rangle$ is called a formal context. For each $A \subseteq X$ and $B \subseteq Y$ denote by $A^{\uparrow}$ a subset of Y and by $B^{\downarrow}$ a subset of X defined by

$$\begin{aligned} A^{\uparrow} &= \{y \in Y \mid \text{for each } x \in A: \ \langle x, y\rangle \in I\},\\ B^{\downarrow} &= \{x \in X \mid \text{for each } y \in B: \ \langle x, y\rangle \in I\}. \end{aligned}$$

That is, $A^{\uparrow}$ is the set of all attributes from Y shared by all objects from A; similarly, $B^{\downarrow}$ is a set of all objects sharing all attributes from B. A formal concept in $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of $A \subseteq X$ and $B \subseteq Y$ satisfying $A^{\uparrow} = B$ and $B^{\downarrow} = A$. That is, a formal concept consists of a set A (so-called extent) of objects which fall under the concept and a set B (so-called intent) of attributes which fall under the concept such that A is the set of all objects sharing all attributes from B and, conversely, B is the collection of all attributes from Y shared by all objects from A. Alternatively, formal concepts can be defined as maximal rectangles (i.e., rectangular subtables of) of $\langle X, Y, I\rangle$ which are full of 1's: Call a pair $\langle A, B\rangle$ of $A \subseteq X$ and $B \subseteq Y$ a rectangle; a rectangle $\langle A, B\rangle$ is contained in $\langle X, Y, I\rangle$ if for the corresponding rectangular relation $A \times B$ we have $A \times B \subseteq I$ where $A \times B$ is a subset of the Cartesian product $X \times Y$, namely, $A \times B = \{\langle x, y\rangle \mid x \in A, y \in B\}$. Then, one can easily see the following "rectangular characterization" of formal concepts.

Theorem 1: $\langle A, B\rangle$ is a formal concept of $\langle X, Y, I\rangle$ iff $\langle A, B\rangle$ is a maximal rectangle contained in $\langle X, Y, I\rangle$ (i.e., $\langle A, B\rangle$ is contained in $\langle X, Y, I\rangle$ and there is no $A' \supset A$ or $B' \supset B$ such that $\langle A', B\rangle$ or $\langle A, B'\rangle$ is contained in $\langle X, Y, I\rangle$). □

A set $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ of all formal concepts in data $\langle X, Y, I\rangle$ can be equipped with a partial order $\leq$ (modeling the subconcept-superconcept hierarchy, e.g. "dog" $\leq$ "mammal") defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1). \quad (2)$$

Note that $^{\uparrow}$ and $^{\downarrow}$ form a so-called Galois connection [8] and that $\mathcal{B}(X, Y, I)$ is in fact a set of all fixed points of $^{\uparrow}$ and $^{\downarrow}$. Under $\leq$, $\mathcal{B}(X, Y, I)$ happens to be a complete lattice, called a concept lattice of $\langle X, Y, I\rangle$, the basic structure of which is described by the so-called main theorem of concept lattices [8]:

Theorem 2: (1) The set $\mathcal{B}(X, Y, I)$ is under $\leq$ a complete lattice where the infima and suprema are given by

$$\begin{aligned} \textstyle\bigwedge_{j\in J}\langle A_j, B_j\rangle &= \langle \textstyle\bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle,\\ \textstyle\bigvee_{j\in J}\langle A_j, B_j\rangle &= \langle (\textstyle\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j\rangle. \end{aligned}$$

(2) Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \leq\rangle$ is isomorphic to $\mathcal{B}(X, Y, I)$ iff there are mappings $\gamma : X \to V$, $\mu : Y \to V$ such that

(i) $\gamma(X)$ is $\bigvee$-dense in V, $\mu(Y)$ is $\bigwedge$-dense in V;
(ii) $\gamma(x) \leq \mu(y)$ iff $\langle x, y\rangle \in I$. □

For a detailed information on formal concept analysis we refer to [6], [8] where a reader can find theoretical foundations, methods and algorithms, and applications in various areas.

III. Formal Concepts as Factors for BFA

The idea of using formal concepts as factors has been presented in [11]. In what follows, we precisely formulate the idea. Suppose I is an $n \times m$ binary matrix. We can think of the matrix as of a data table $\langle X, Y, I\rangle$ with $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$, and with $\langle x_i, y_j\rangle \in I$ iff $I_{ij} = 1$ (I_{ij} is the entry at the i-th row and j-th column of matrix I). Notice our slight abuse of notation: I denotes both the binary matrix and the relation between X and Y (there is, however, no danger of confusion). Consider now the associated concept lattice $\mathcal{B}(X, Y, I)$. The question is: Is there a set $\mathcal{F} \subseteq \mathcal{B}(X, Y, I)$ of formal concepts which are suitable as factors for decomposing I into A and B? More precisely, let

$$\mathcal{F} = \{\langle A_1, B_1\rangle, \dots, \langle A_k, B_k\rangle\} \subseteq \mathcal{B}(X, Y, I). \qquad (3)$$

Consider the matrices A and B defined as follows. A is an $n \times k$ binary matrix with entries A_{il} ($1 \le i \le n$, $1 \le l \le k$) defined by

$$A_{il} = \begin{cases} 1 & \text{if } x_i \in A_l, \\ 0 & \text{if } x_i \notin A_l; \end{cases} \qquad (4)$$

B is an $k \times m$ binary matrix with entries B_{lj} ($1 \le l \le k$, $1 \le j \le m$) defined by

$$B_{lj} = \begin{cases} 1 & \text{if } y_j \in B_l, \\ 0 & \text{if } y_j \notin B_l. \end{cases} \qquad (5)$$

That is, columns of A correspond to (characteristic vectors of) extents A_l of the formal concepts $\langle A_l, B_l\rangle$, and rows of B correspond to (characteristic vectors of) intents B_l of the formal concepts $\langle A_l, B_l\rangle$. $\mathcal{F}$ is then suitable as a set of factors if $I = A \circ B$ (I is equal to the Boolean matrix product of A and B) or $I \approx A \circ B$ (I is approximately equal to $A \circ B$, with "$\approx$" appropriately defined).

A. Main results on BFA with formal concepts as factors

In what follows, we denote the matrices A and B induced by a set $\mathcal{F}$ by (4) and (5) by $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$.

Remark 1: Note that, in fact, the numbering of the concepts in $\mathcal{F}$ does not matter (if renumbering the concepts in $\mathcal{F}$ leads to a corresponding rearrangement of columns in matrix A and rearrangement of rows in matrix B; but this plays no role in whether I can be decomposed by means of a product of A and B). Nevertheless, it might be reasonable to consider an ordering of factors in $\mathcal{F}$ with $\langle A_1, B_1\rangle$ being the most important factor (in some sense), which is similar to the case of classical factor analysis. We do not consider the issue of importance of factors in this paper. □

We first consider a general problem of factorizability of I.

Definition 1: An $n \times m$ binary matrix I is called *factorizable* if there is an $n \times k$ binary matrix A and a $k \times m$ binary matrix B such that $I = A \circ B$ with $A \circ B$ defined by (1).

The k elements (corresponding to columns of A and rows of B) are called *factors* of I.

Definition 2: An $n \times m$ binary matrix I is called *concept-factorizable* if there is a set $\mathcal{F}$ of formal concepts of $\mathcal{B}(X, Y, I)$ of the form (3) such that for the corresponding matrices $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ we have $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$.

Therefore, if I is concept-factorizable with $\mathcal{F}$, we call the formal concepts of $\mathcal{F}$ factors (or factor concepts). We need the following lemma

Lemma 1 (product $A_{\mathcal{F}} \circ B_{\mathcal{F}}$ as union of rectangles): Let $\mathcal{F} = \{\langle A_l, B_l\rangle \mid 1 \le l \le k, A_l \subseteq X, B_l \subseteq Y\}$ (i.e., A_l and B_l are subsets of X and Y, $\langle A_l, B_l\rangle$ need not be formal concepts), let $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$. Then $I_{ij} = 1$ ($1 \le i \le n$, $1 \le j \le m$) iff there exists l such that $x_i \in A_l$ and $y_j \in B_l$.

Proof: By definition of I, we have $I_{ij} = \bigvee_{l=1}^{k} A_{il} \otimes B_{lj}$. Therefore, $I_{ij} = 1$ iff $\bigvee_{l=1}^{k} A_{il} \otimes B_{lj} = 1$ iff there exists l such that $A_{il} = 1$ and $B_{lj} = 1$ iff there exists l such that $x_i \in A_l$ and $y_j \in B_l$. ■

Remark 2: (1) Notice the following meaning of Lemma 1. Given a set $\mathcal{F}$ of pairs $\langle A_l, B_l\rangle$ of subsets of X and Y, the relation I corresponding to $A_{\mathcal{F}} \circ B_{\mathcal{F}}$ is constructed as follows: I is a union of rectangular relations $A_l \times B_l$ corresponding to rectangles $\langle A_l, B_l\rangle$, i.e.

$$I = A_1 \times B_1 \;\cup\; \cdots \;\cup\; A_k \times B_k. \qquad (6)$$

(2) Conversely, given an $n \times k$ binary matrix A and a $k \times m$ binary matrix B, we can consider a set $\mathcal{F} = \{\langle A_l, B_l\rangle \mid 1 \le l \le k, A_l \subseteq X, B_l \subseteq Y\}$ with A_l and B_l defined as follows:

$$x_i \in A_l \text{ iff } A_{il} = 1, \quad y_j \in B_l \text{ iff } B_{lj} = 1. \qquad (7)$$

Then, of course, $A = A_{\mathcal{F}}$ and $B = B_{\mathcal{F}}$. Therefore, according to (1), the binary relation I corresponding to the product $A \circ B$ is a union of rectangles $A_l \times B_l$, as in (6). □

Example 1: Let $n = 4$, $m = 6$, $k = 3$, $A_1 = \{x_2, x_3\}$, $B_1 = \{y_4, y_5, y_6\}$, $A_2 = \{x_1, x_2\}$, $B_2 = \{y_3, y_4\}$, $A_3 = \{x_3\}$, $B_3 = \{y_2, y_3, y_4\}$. For $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ we have

$$A_{\mathcal{F}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{\mathcal{F}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

According to Lemma 1 and Remark 2, for the relation I corresponding to $A_{\mathcal{F}} \circ B_{\mathcal{F}}$ we have $I = \bigcup_{l=1}^{3} A_l \times B_l$. In a matrix setting, this means that $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$ is built as a union of binary matrices C_1, C_2, C_3 where C_l corresponds to the rectangular relation $A_l \times B_l$. We have

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

The following theorem shows that each I is concept-factorizable. In fact, one can take $\mathcal{F} = \mathcal{B}(X, Y, I)$, i.e. the set of all formal concepts of $\mathcal{B}(X, Y, I)$.

Theorem 3 (concept-factorizability): For $\mathcal{F} = \mathcal{B}(X, Y, I)$ we have $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$. Therefore, each binary matrix I is concept-factorizable.

Proof: Denote $D = A_{\mathcal{F}} \circ B_{\mathcal{F}}$. By Lemma 1 we have $D_{ij} = 1$ iff there exists a formal concept $\langle A_l, B_l\rangle \in \mathcal{B}(X, Y, I)$ such that $x_i \in A_l$ and $y_j \in B_l$. Therefore, we have to check that $\langle x_i, y_j\rangle \in I$ iff there exists $\langle A_l, B_l\rangle \in \mathcal{B}(X, Y, I)$

such that $x_i \in A_l$ and $y_j \in B_l$ which is easy. On the one hand, if $\langle x_i, y_j\rangle \in I$ then for $\langle A_l, B_l\rangle = \langle \{x_i\}^{\uparrow\downarrow}, \{x_i\}^{\uparrow}\rangle \in \mathcal{B}(X,Y,I)$ we have both $x_i \in A_l$ and $y_j \in B_l$. On the other hand, if $x_i \in A_l$ and $y_j \in B_l$ for some $\langle A_l, B_l\rangle \in \mathcal{B}(X,Y,I)$, then since $B_l = A_l^{\uparrow}$, the definition of $^{\uparrow}$ yields $\langle x_i, y_j\rangle \in I$. ■

Corollary 1: For $\mathcal{F} \subseteq \mathcal{B}(X,Y,I)$ we have $A_{\mathcal{F}} \circ B_{\mathcal{F}} \subseteq I$, i.e. $(A_{\mathcal{F}} \circ B_{\mathcal{F}})_{ij} \leq I_{ij}$ for each i and j ($1 \leq i \leq n, 1 \leq j \leq m$).

Although Theorem 3 gives a way to factorize I, the obvious disadvantage is that the number $|\mathcal{B}(X,Y,I)|$ of all concepts of $\mathcal{B}(X,Y,I)$ (and thus the number of factors suggested by Theorem 3) is usually much larger than m, i.e. than the number of original attributes. Before we show that one can, in fact, select a smaller set of factor concepts, we show that the general factorizability implies concept-factorizability with the number of factor concepts being less or equal to the number of the original factors.

Theorem 4 (factorizability ⇒ concept-factorizability): Let I be factorizable with k factors, i.e. $I = A \circ B$ where A and B are $n \times k$ and $k \times m$ binary matrices, respectively. Then I is concept-factorizable with at most k concept factors, i.e. there is $\mathcal{F} \subseteq \mathcal{B}(X,Y,I)$ with $|\mathcal{F}| \leq k$ such that $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$.

Proof: Sketch (details to be presented in full version): Consider the pairs $\langle A_l, B_l\rangle$ ($1 \leq l \leq k$) defined by (7). The proof is based on the fact that for each Therefore, for each $\langle A_l, B_l\rangle$ ($1 \leq l \leq k$) there exists $\langle C_l, D_l\rangle \in \mathcal{B}(X,Y,I)$ such that $A_l \subseteq C_l$ and $B_l \subseteq D_l$. ■

Remark 3: Note that, under the notation from the proof of Theorem 4, we might have $|\mathcal{F}| < k$ (the number of concept factors may be strictly smaller than the number of the original factors). Namely, it may happen that two different rectangles $\langle A_l, B_l\rangle$ and $\langle A_{l'}, B_{l'}\rangle$ are contained in the same maximal rectangle, i.e. one can take $\langle C_l, D_l\rangle = \langle C_{l'}, D_{l'}\rangle$.

In fact, as shown by Theorem 4, if the number of factors is taken as a quality of factorization, concept-factorizability is as good as factorizability, i.e. concept-factorization is the best way to factorize.

In the next part, we will make use of the following two sets of formal concepts. Let I be a given binary matrix. Consider the following two sets of formal concepts of the corresponding $\langle X, Y, I\rangle$. First, formal concepts of

$$\mathcal{O}(X,Y,I) = \{\langle \{x_i\}^{\uparrow\downarrow}, \{x_i\}^{\uparrow}\rangle \mid 1 \leq i \leq n\} \quad (8)$$

are called object-concepts (they correspond to rows of I in that the rows of I are the intents of object concepts). Second, formal concepts of

$$\mathcal{A}(X,Y,I) = \{\langle \{y_j\}^{\downarrow}, \{y_j\}^{\downarrow\uparrow}\rangle \mid 1 \leq j \leq m\} \quad (9)$$

are called attribute-concepts (they correspond to columns of I in that the columns of I are the extents of attribute-concepts).

The following theorem shows two small sets of concept factors.

Theorem 5 (factorizability by object/attribute concepts): (1) Let $\mathcal{F} = \mathcal{O}(X,Y,I)$. Then I is concept-factorizable with $\mathcal{F}$ being the set of factor concepts.

(2) Let $\mathcal{F} = \mathcal{A}(X,Y,I)$. Then I is concept-factorizable with $\mathcal{F}$ being the set of factor concepts.

Proof: The proof follows easily from the fact that $\langle x_i, y_j\rangle \in I$ iff for $\langle A_l, B_l\rangle = \langle \{x_i\}^{\uparrow\downarrow}, \{x_i\}^{\uparrow}\rangle$ we have $x_i \in A_l$ and $y_j \in B_l$. Therefore, Lemma 1 yields that I is concept-factorizable with $\mathcal{F} = \mathcal{O}(X,Y,I)$. The proof for $\mathcal{F} = \mathcal{A}(X,Y,I)$ is analogous. ■

Therefore, according to Theorem 5, each I is concept-factorizable with $\mathcal{F}$ containing at most $\min(n,m)$ concept factors.

The next theorem describes mandatory formal concepts, i.e. factor concepts which belong to each set $\mathcal{F}$ of factor concepts for a given I.

Theorem 6 (mandatory factor concepts): If I is concept-factorizable with a set $\mathcal{F}$ of factor concepts then $\mathcal{O}(X,Y,I) \cap \mathcal{A}(X,Y,I) \subseteq \mathcal{F}$.

Proof: The assertion follows from the fact that if $\langle A, B\rangle \in \mathcal{O}(X,Y,I) \cap \mathcal{A}(X,Y,I)$, e.g., $\langle A, B\rangle = \langle \{y_j\}^{\downarrow}, \{x_i\}^{\uparrow}\rangle$, then $\langle A, B\rangle$ is the only formal concept from $\mathcal{B}(X,Y,I)$ such that $\langle x_i, y_j\rangle \in \langle A, B\rangle$. Therefore, according to Lemma 1 if $\langle A, B\rangle \notin \mathcal{F}$ then for $I' = A_{\mathcal{F}} \circ B_{\mathcal{F}}$ we have $I'_{ij} = 0$, contrary to $I_{ij} = 1$. ■

Remark 4: It is easy to see that for some I, the smallest set $\mathcal{F}$ of factor concepts contains $|Y|$ attributes, i.e. in bad cases, the factorization does not lead to dimension reduction (details omitted).

B. Algorithms

In this section, we discuss the problem of generating a good set of factor concepts for a given binary matrix I. Due to lack of space, a detailed description of algorithms and experimental results will be presented in a forthcoming paper.

The basic observation is that according to previous results, finding a set $\mathcal{F}$ of factor concepts such that $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$ can be reduced to the well-known set-covering problem. Importantly, there are good approximation algorithms for the set-covering problem, see e.g. [7], i.e. one can efficiently find almost minimal sets of factors concepts. In addition to that, we can make advantage of a special nature of our problem and of the above-presented theoretical insight. For instance, Theorem 6 implies that we can reduce the search space by removing the mandatory factor concepts. Note also that $\mathcal{B}(X,Y,I)$, i.e. the set of candidate factor concepts, can be efficiently computed (for instance, using NextClosure algorithm [8]).

C. Examples

In this section, we present illustrative examples on BFA. Our main aim is to illustrate the notions and results introduced in previous sections.

Suppose we have a record of a collection of patients. For each of the patients, our record contains set of symptoms. For our purpose, we consider a set Y of 8 symptoms which we denote by $y_1, \ldots, y_8$ (i.e., $Y = \{y_1, \ldots, y_8\}$). The symptoms are described in Tab. I.

Suppose our collection of patients contains 12 patients. We denote the i-th patient by x_i and put $X = \{x_1, \ldots, x_{12}\}$.

TABLE I

SYMPTOMS AND THEIR DESCRIPTIONS

symptom	symptom description
y_1	headache
y_2	fever
y_3	painful limbs
y_4	swollen glands in neck
y_5	cold
y_6	stiff neck
y_7	rash
y_8	vomiting

TABLE II

FORMAL CONCEPTS OF DATA GIVEN BY PATIENTS AND THEIR SYMPTOMS

c_i	$\langle A_i, B_i \rangle$
c_0	$\langle \{\}, \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\} \rangle$
c_1	$\langle \{x_1, x_5, x_9, x_{11}\}, \{y_1, y_2, y_3, y_5\} \rangle$
c_2	$\langle \{x_2, x_4, x_{12}\}, \{y_1, y_2, y_6, y_8\} \rangle$
c_3	$\langle \{x_3, x_6, x_7\}, \{y_2, y_5, y_7\} \rangle$
c_4	$\langle \{x_3, x_6, x_7, x_8, x_{10}\}, \{y_7\} \rangle$
c_5	$\langle \{x_1, x_3, x_5, x_6, x_7, x_9, x_{11}\}, \{y_2, y_5\} \rangle$
c_6	$\langle \{x_1, x_2, x_4, x_5, x_9, x_{11}, x_{12}\}, \{y_1, y_2\} \rangle$
c_7	$\langle \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{11}, x_{12}\}, \{y_2\} \rangle$
c_8	$\langle \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}, \{\} \rangle$

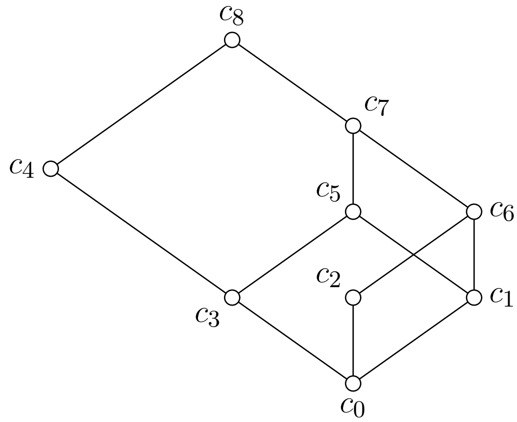

Fig. 1. Hase diagram of concept lattice given by patients and their symptoms

TABLE III

DISEASES AND THEIR SYMPTOMS (AN EXCERPT)

disease	symptoms
chickenpox	rash
flu	headache, fever, painful limbs, cold
measles	fever, cold, rash
meningitis	headache, fever, stiff neck, vomiting
⋮	⋮

Suppose our record is described by a 12×8 binary matrix I given by

$$I = \begin{pmatrix} 1&1&1&0&1&0&0&0 \\ 1&1&0&0&0&1&0&1 \\ 0&1&0&0&1&0&1&0 \\ 1&1&0&0&0&1&0&1 \\ 1&1&1&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&0&1&0&1&0 \\ 0&0&0&0&0&0&1&0 \\ 1&1&1&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 1&1&1&0&1&0&0&0 \\ 1&1&0&0&0&1&0&1 \end{pmatrix}.$$

That is, rows correspond to patients, columns correspond to symptoms, and I_{ij} (entry at i-th row and j-th column) is equal to 1 if patient x_i has symptom y_j and is equal to 0 if patient x_i does not have symptom y_j. Matrix I can be seen as follows: patients are represented in an 8-dimensional space of attributes.

Our intention is to find a set $\mathcal{F}$ of factor concepts. That is, we want to find some $\mathcal{F} \subseteq \mathcal{B}(X, Y, I)$ such that $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$. Note that we use I to denote both the 12×8 matrix and the corresponding relation between X and Y. Note also that $\langle x_i, y_j \rangle \in I$ iff $I_{ij} = 1$.

Let us first look at the concept lattice $\mathcal{B}(X, Y, I)$. $\mathcal{B}(X, Y, I)$ contains 9 formal concepts. That is $\mathcal{B}(X, Y, I) = \{c_0, \dots, c_8\}$ and each c_i is of the form $c_i = \langle A_i, B_i \rangle$ where $A_i \subseteq X$ is a set of patients, $B_i \subseteq Y$ is a set of symptoms such that $A_i^{\uparrow} = B_i$ and $B_i^{\downarrow} = A_i$ (i.e., B_i is the set of all symptoms common to all patients from A_i, and A_i is the set of all patients sharing all symptoms from B_i). All formal concepts from $\mathcal{B}(X, Y, I)$ are depicted in Tab. II.

For instance, the extent A_3 of formal concept c_3 consists of patients x_3, x_6, x_7, and the intent B_3 of c_3 consists of attributes y_2, y_5, y_7. Note that since the concept lattice $\mathcal{B}(X, Y, I)$ equipped with a partial order $\leq$ (the subconcept-superconcept hierarchy) can be visualized by its Hasse diagram. The Hasse diagram of $\mathcal{B}(X, Y, I)$ is shown in Fig. 1. We can see from the diagram that, e.g., $c_3 \leq c_4$, i.e., formal concept c_4 is more general than formal concept c_3. This is because the extent A_3 of c_3 is contained in the extent A_4 of c_4, i.e. $A_3 \subseteq A_4$, meaning that each patient covered by c_3 is also covered by c_4. Equivalently, the extent B_4 of c_4 is contained in the extent B_3 of c_3, i.e. $B_4 \subseteq B_3$, meaning that each attribute characteristic for c_4 is also a characteristic attribute of c_3. In a similar way, one can see that $c_3 \leq c_8$, $c_2 \leq c_7$, etc.

Let us now look at the meaning of formal concepts $c_0, \dots, c_8$ from $\mathcal{B}(X, Y, I)$. One naturally expects that the concepts, which are given by groups of patients (concept extents) and groups of symptoms (concept intents) will be related to diseases. This is, indeed the case. Tab. III shows an excerpt from a family medical book. We can see that, e.g., formal concept c_1 represents flu because c_1 covers just headache, fever, painful limbs, cold, which are exactly the characteristic attributes of flu. In the same way, c_2, c_3, and c_4 represent meningitis, measles, and chickenpox. However, $\mathcal{B}(X, Y, I)$ contains also formal concepts which can be interpreted as "suspicion of (a disease)". For instance, c_5 can be interpreted as "suspicion of flu or measles" because the intent B_5 of c_5 contains attributes y_2 (fever) and y_5 (cold) which belong to the characteristic attributes of both flu and measles. Note that $\mathcal{B}(X, Y, I)$ also contains an "empty concept" c_0 (the concept is the least concept in the conceptual hierarchy and applies to no patient) and the universal concept c_8 (applies to all patients). Both the empty and the universal concepts are usually not interesting. Verbal descriptions of formal concepts from $c_0, \dots, c_8$ are presented in Tab. IV. Note also that the verbal description and the conceptual hierarchy are compatible. For instance, we have $c_1 \leq c_6$, i.e. c_6 represents a weaker concept (more general concept) than c_1, which corresponds to their descriptions: c_1 is "flu", c_6 is "suspicion of flu or meningitis".

TABLE IV

DESCRIPTION OF FORMAL CONCEPTS OF DATA GIVEN BY PATIENTS AND THEIR SYMPTOMS

formal concept	concept description
c_0	empty concept
c_1	"flu"
c_2	"meningitis"
c_3	"measles"
c_4	"chickenpox"
c_5	"suspicion of flu or measles"
c_6	"suspicion of flu or meningitis"
c_7	"suspicion of flu or measles or meningitis"
c_8	universal concept

Let us now look at the factorization of I. Our main aim is to illustrate the above results. According to Theorem 1, we can factorize I by $\mathcal{F} = \mathcal{B}(X, Y, I)$. However, since $|\mathcal{F}| = 9 > 8 = |Y|$, this would mean that we would end up with a number of factors bigger than the number of the original attributes. Therefore, we are looking for some small set $\mathcal{F} \subset \mathcal{B}(X, Y, I)$ of factor concepts. First, it can be easily shown that the empty concept and the universal concept can always be disregarded. Furthermore, $\mathcal{B}(X, Y, I)$ contains the following object- and attribute-concepts: $\mathcal{O}(X, Y, I) = \{c_1, c_2, c_3, c_4\}$, $\mathcal{A}(X, Y, I) = \{c_0, c_1, c_2, c_4, c_5, c_6, c_7\}$. Thus, $\mathcal{O}(X, Y, I) \cap \mathcal{A}(X, Y, I) = \{c_1, c_2, c_4\}$. According to Theorem 6, $\mathcal{F}$ needs to contain $\mathcal{O}(X, Y, I) \cap \mathcal{A}(X, Y, I) = \{c_1, c_2, c_4\}$. Then, $\mathcal{F}'' = \{c_1, c_2, c_4\}$ is almost a set of factor concepts. Namely, for $I'' = A_{\mathcal{F}''} \circ B_{\mathcal{F}''}$, I'' differs from I only in that $I''_{3,5} = I''_{6,5} = I''_{7,5} = 0$ while $I_{3,5} = I_{6,5} = I_{7,5} = 1$. From Fig. 1 and Theorem 2, we can directly see that the only formal concepts which cover $\langle x_3, y_5\rangle$, $\langle x_6, y_5\rangle$, and $\langle x_7, y_5\rangle$, are c_3 and c_5. Therefore, according to Lemma 1, there are just two minimal sets of factor concepts, namely

$$\mathcal{F} = \{c_1, c_2, c_3, c_4\} \quad \text{and} \quad \mathcal{F}' = \{c_1, c_2, c_4, c_5\}.$$

Thus, I can be decomposed by $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$ or $I = A_{\mathcal{F}'} \circ B_{\mathcal{F}'}$. For instance, we have

$$A_{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_{\mathcal{F}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and similarly for $A_{\mathcal{F}'}$ and $B_{\mathcal{F}'}$. From the point of view of dimension reduction, instead in an 8-dimensional space of symptoms (as described by I), the patients are now described in an 4-dimensional space of disease-like concepts (as described by $A_{\mathcal{F}}$).

IV. FURTHER REMARKS AND FURTHER RESEARCH

Future research will focus on the following topics:

- Approximate factorizability. Extension of the results so that one requires a weaker condition I approximately equals $A_{\mathcal{F}} \circ B_{\mathcal{F}}$. The main idea is to omit or glue together factor concepts.
- Extension to many-valued matrices. It seems very interesting from the application point of view to extend BFA for the case when I contains degrees, e.g. reals from $[0, 1]$ instead just binary values 0 and 1. Using available results on FCA of data with fuzzy attributes, see [3], [4], [5], we obtained results analogous to those presented in this paper. Basically, the factors in case when I contains degrees are so-called formal fuzzy concepts which are the fixed points of fuzzy Galois connections.
- Algorithms. In Section III-B, we provided basic hint to the reduction of the problem of finding a small set of factor concepts to the set-covering problem. Our results enable us to speed up the available algorithms (removing mandatory concepts from the search space). The main goal is look for further results with a speed-up ramifications. In addition to that, several heuristic approaches need to be tested (details omitted).
- Comparison with approaches to BFA using Hopfield neural nets [15], [17]. In addition to that, there is an interpretation of concept lattices in terms of so-called bidirectional associative memory which is a Hopfield-like neural net, see [2]. This might provide interesting connections between BFA using Hopfield-like neural nets and BFA using formal concepts.

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