Compositions of Fuzzy Relations With Hedges II

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***Abstract*— We continue our previous paper and explore an extension of compositions of fuzzy relations. The extension consists in modifying the standard formulas for composition by particular unary functions on the set of truth degrees. We study fuzzy relational equations using the new composition operators. As a particular example, the new operators lead to a decomposition of a fuzzy relation into a crisp relation and a fuzzy relation. We provide solvability criteria for several types of fuzzy relational equations. In addition, we present a connection between the new compositions and a particular type of data dependencies.**

I. Introduction

Fuzzy relations and their calculus are employed virtually in every application of fuzzy logic. In our previous paper [4], we introduced and studied a generalization of ordinary compositions of fuzzy relations. The generalization consists in inserting a particular unary function, the so-called truth-stressing hedge, on the set of truth degrees in the standard definition of compositions. In this paper, we explore fuzzy relational equations with the new composition of fuzzy relations. An interesting fact is that a particular example of our approach leads to decompositions of a fuzzy relation into a crisp relation and a fuzzy relation. In addition, we present a connection of the new composition to fuzzy attribute implications which are particular data dependencies.

Let us present a particular application of the results developed in this paper. Suppose one has two fuzzy relations: (i) a fuzzy relation describing performance of individual workers in terms of their speed, accuracy, punctuality, etc.; (ii) a fuzzy relation describing performance of workgroups in the same terms. If the structure of workgroups is not known, i.e., if we do not know the relation between individual workers and work groups, we can reveal presence/absence of individual workers in groups by finding a crisp solution to a fuzzy relational equation. This is a particular application of decompositions of a fuzzy relation (workgroups × performance) into a crisp relation (individuals × workgroups) and a fuzzy relation (individuals × performance). Clearly, a crisp solution is needed because a worker either belongs to a group or not.

II. Preliminaries

In this section we present an overview of notions of fuzzy logic and fuzzy set theory we will be using in this paper. Details can be found e.g. in [5], [16], [18], a good introduction to fuzzy logic and fuzzy sets is presented in [20].

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A. Complete Residuated Lattices

Our approach to fuzzy sets and fuzzy relations is based on complete residuated lattices which are used as basic structures of truth degrees. A complete residuated lattice [5], [18] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. That fact that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice means that general infima $\bigwedge_{i\in I} a_i$ and general suprema $\bigvee_{i\in I} a_i$ exist for any subset $\{a_i \,|\, i \in I\} \subseteq L$. As it is usual in the context of fuzzy logic, elements $a \in L$ are called truth degrees. Operations $\otimes$ and $\rightarrow$ are truth functions of logical connectives "fuzzy conjunction" (also called "multiplication") and "fuzzy implication" (also called "residuum"). We denote by $\leq$ the lattice order induced by $\mathbf{L}$. Using the adjointness property $a \leq b$ iff $a \rightarrow b = 1$. Complete residuated lattice $\mathbf{L}$ is called linearly ordered (or a chain) if, for each $a, b \in L$, $a \leq b$ or $b \leq a$.

The most important complete residuated lattices are those defined on the real unit interval. In such a case, $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz: $a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$; Gödel (minimum): $a \otimes b = a \wedge b$, $a \rightarrow b = b$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$; Goguen (product): $a \otimes b = a \cdot b$, $a \rightarrow b = \frac{b}{a}$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$.

A special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic.

Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

B. Truth-Stressing Hedges

Complete residuated lattices can be equipped with additional fundamental operations. In this paper, we are going to use particular unary operations called truth-stressing hedges. An idempotent truth-stressing hedge (shortly, a hedge) on a complete residuated lattice $\mathbf{L}$ is a mapping ${}^* : L \rightarrow L$ satisfying the following conditions

$$1^* = 1, \tag{1}$$

$$a^* \leq a, \tag{2}$$

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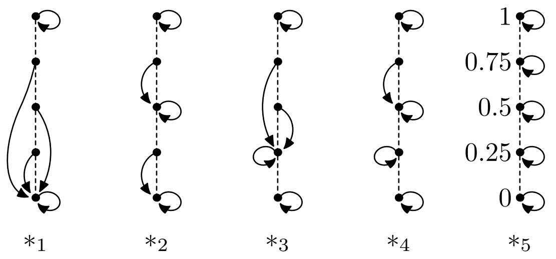

Fig. 1. Hedges on a five-element Łukasiewicz chain

$$(a \to b)^* \leq a^* \to b^*, \tag{3}$$

$$a^{**} = a^*, \tag{4}$$

for each $a, b \in L$. Truth-stressing hedges were investigated from the point of view of fuzzy logic in narrow sense by Hájek [19] who showed complete axiomatizations of BL-logics equipped with unary connectives "very true". Note that [19] does not use (4) but uses $(a \vee b)^* \leq a^* \vee b^*$ which we do not use. Let us note that properties (2)–(4) have natural interpretations. For instance, (2), called subdiagonality, can be read: "if a is very true, then a is true"; (3) can be read: "if $a \to b$ is very true and if a is very true, then b is very true", etc. Truth-stressing hedges are monotone, i.e.

$$a \leq b \text{ implies } a^* \leq b^*. \tag{5}$$

On each complete residuated lattice $\mathbf{L}$, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a$ ($a \in L$);

(ii) globalization [22], i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Note that globalization agrees with Baaz's Δ-operation [1] in case of linearly ordered residuated lattices.

Example 1: Let $\mathbf{L}$ be a finite residuated lattice with $L = \{0, 0.25, 0.5, 0.75, 1\}$, $\wedge$ and $\vee$ being minimum and maximum, respectively, $\otimes$ be the Łukasiewicz conjunction on L with its residuum $\to$. There are five idempotent truth-stressing hedges on $\mathbf{L}$. The hedges are depicted by their diagrams in Fig. 1. The left-most hedge $*_1$ is globalization, see (6). On the contrary, the right-most hedge $*_5$ is identity. There are three intermediate hedges $*_2$, $*_3$, and $*_4$. For instance, in case of $*_2$, we have $0^{*_2} = 0.25^{*_2} = 0$, $0.5^{*_2} = 0.75^{*_2} = 0.5$, and $1^{*_2} = 1$.

Example 2: There is just one hedge on the two-element Boolean algebra $\mathbf{2}$. Namely, hedge * such that $0^* = 0$ and $1^* = 1$. Hence, in case of the two-element Boolean algebra, globalization coincides with identity.

C. Fuzzy Sets and Fuzzy Relations

Suppose that $\mathbf{L}$ is our structure of truth degrees. We define usual notions: a fuzzy set (an $\mathbf{L}$-set) A in universe U is a mapping $A : U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Operations with fuzzy sets are defined componentwise. For instance, the intersection of fuzzy sets $A, B \in \mathbf{L}^U$ is a fuzzy set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. A binary fuzzy relation (a binary $\mathbf{L}$-relation) I between X and Y is a mapping $I : X \times Y \to L$, $I(x, y)$ being interpreted as "the degree to which x and y are related by I". By definition, a binary fuzzy relation $I : X \times Y \to L$ is a fuzzy set in the universe $X \times Y$, i.e. $I \in \mathbf{L}^{X \times Y}$.

In the following we use well-known properties of residuated lattices, fuzzy sets, and fuzzy relations which can be found in monographs [5], [16], [18], [20].

III. Compositions of Fuzzy Relations with Hedges

We start by recalling the definition of hedge-based compositions of fuzzy relations which we introduced in [4]. We assume that R and S are $\mathbf{L}$-relations between X and Y, and Y and Z, respectively. Recall first the definition of $\circ$-composition, $\triangleleft$-composition, and $\triangleright$-composition of fuzzy relations, see e.g. [2], [3], [5]:

$$(R \circ S)(x, z) = \textstyle\bigvee_{y \in Y} \big(R(x, y) \otimes S(y, z)\big), \tag{7}$$

$$(R \triangleleft S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(R(x, y) \to S(y, z)\big), \tag{8}$$

$$(R \triangleright S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(S(y, z) \to R(x, y)\big), \tag{9}$$

for all $x \in X$, $z \in Z$. $(R \circ S)(x, z)$ is the truth degree of proposition "there is $y \in Y$ such that $\langle x, y \rangle$ is in R and $\langle y, z \rangle$ is in S". $(R \triangleleft S)(x, z)$ is the truth degree of proposition "for every $y \in Y$: if $\langle x, y \rangle$ is in R then $\langle y, z \rangle$ is in S". $(R \triangleright S)(x, z)$ is the truth degree of proposition "for every $y \in Y$: if $\langle y, z \rangle$ is in S then $\langle x, y \rangle$ is in R". The following definition presents our extension.

Definition 1 ([4]): Let R and S be $\mathbf{L}$-fuzzy relations between X and Y and between Y and Z. Fuzzy relations $(R \circ^* S)$, $(R \triangleleft^* S)$, and $(R \triangleright^* S)$ between X and Z are defined by

$$(R \circ^* S)(x, z) = \textstyle\bigvee_{y \in Y} \big(R(x, y)^* \otimes S(y, z)\big), \tag{10}$$

$$(R \triangleleft^* S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(R(x, y)^* \to S(y, z)\big), \tag{11}$$

$$(R \triangleright^* S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(S(y, z)^* \to R(x, y)\big), \tag{12}$$

for all $x \in X$, $z \in Z$.

$(R \circ^* S)(x, z)$ is the truth degree of proposition "there is $y \in Y$ such that it is very true that $\langle x, y \rangle$ is in R and $\langle y, z \rangle$ is in S". $(R \triangleleft^* S)(x, z)$ is the truth degree of proposition "for every $y \in Y$: if it is very true that $\langle x, y \rangle$ is in R then $\langle y, z \rangle$ is in S". $(R \triangleright^* S)(x, z)$ is the truth degree of proposition "for every $y \in Y$: if it is very true that $\langle y, z \rangle$ is in S then $\langle x, y \rangle$ is in R". If * is the identity, (10), (11), and (12) become (7), (8), and (9). Therefore, the new compositions generalize the ordinary ones.

IV. Relational Equations With Hedges

Fuzzy relational equations represent a classic topic which was initiated in [21] and is now well covered e.g. in [17], [20]. The topic of relational equations with hedges was started in [4]. Here we present further results. Compared to [4], we consider additional types of fuzzy relational equations. Moreover, we consider more general types of hedges.

Namely, we consider functions $*: L \to L$ satisfying (1), (2), (4), and (5). In [4], we considered (3) instead of (5). Note that (5) follows from (1) and (3). Indeed, if $a \leq b$, then $1 = 1^* = (a \to b)^* \leq a^* \to b^*$, i.e. $a^* \leq b^*$. On the other hand, there exist functions $*$ satisfying (1), (2), (4), and (5), but not (3).

The reason we consider more general functions $*$ in this paper can be articulated as follows. First, the results we are interested in can be proved for the more general setting. Second and more importantly: As shown in [4] and as we will show in more detail in this paper, if a fuzzy relational equation $U \circ^* S = T$ has a solution, it has a solution U for which all degrees $U(x, y)$ are the fixpoints of $*$. Therefore, if $*$ is globalization, which has only 0 and 1 as its fixpoints, we conclude that if $U \circ^* S = T$ has a solution then it has a crisp solution, i.e. there is a crisp relation U which is a solution. Therefore, introducing hedges leads to looking for constrained solutions of fuzzy relational equations. Crispness of fuzzy relations is an example of such constraint. More generally, a user may ask whether there is a solution U to $U \circ^* S = T$ for which $U(x, y) \in K$ for every $x \in X$, $y \in Y$ where K is a suitable subset of L. If $K = \{0, 1\}$, the user wants crisp solutions. If $K = \{0, 0.5, 1\}$, the user wants solutions which use three degrees, etc. If we want to use hedge-based compositions to account for such constraints represented by K, we need to find a hedge for which $\text{fix}(*) = K$. Note that $\text{fix}(*) = \{a \in L \,|\, a^* = a\}$ denotes the set of all fixpoints of $*$. Now, if we use the more general hedges satisfying (1), (2), (4), and (5), we have:

Lemma 1: $K = \text{fix}(*)$ for some function $*: L \to L$ satisfying (1), (2), (4), and (5), iff K is closed w.r.t. arbitrary suprema and $1 \in K$.

Proof: Clearly, (1), (2), (4), and (5) mean that $*$ is an interior operator on the complete lattice $\langle L, \leq \rangle$ for which 1 is an open element. It is a well-known fact that a subset $K \subseteq L$ is the set of fixpoints of an interior operator on L iff K is closed w.r.t. arbitrary suprema. The assertion then readily follows. ■

For the practically important case when $\mathbf{L}$ is a chain and K is finite we get:

Corollary 1: Let $\mathbf{L}$ be a chain, K be a finite subset of L. Then $K = \text{fix}(*)$ for some function $*: L \to L$ satisfying (1), (2), (4), and (5), iff $1 \in K$.

Therefore, using hedges satisfying (1), (2), (4), and (5) is convenient from the user's point of view since in practical situations, these conditions do not impose any serious constraints on K.

We now consider the problem of fuzzy relational equations [17]. The problem can be described as: Given R and T, determine S for which

$$R \odot S \text{ is (at least approximately) equal to } T. \qquad (13)$$

Alternatively, given S and T, determine R. We denote by

$$U \odot S = T, \quad R \odot U = T$$

the fuzzy relational equation with unknown fuzzy relation U. Now, due to limited scope, we present the criteria of solvability of fuzzy relational equations for the particular case where exact equality is required in (13).

First, we need the following lemma which shows the criteria of solvability of fuzzy relational equations with $\circ$-composition. The result is well-known, see [17] for $L = [0, 1]$ and [5] for the general case.

Lemma 2: A fuzzy relational equation $U \circ S = T$ has a solution iff $(S \triangleleft T^{-1})^{-1}$ is a solution.

Theorem 3: A fuzzy relational equation $R \circ^* U = T$ has a solution iff $(R^*)^{-1} \triangleleft T$ is a solution.

Proof: Just check the proof of Theorem 9 in [4] (it is valid even with the weaker hedge). ■

For fuzzy relational equation $U \circ^* S = T$ we start with the following result which immediately follows from definitions.

Theorem 4 ([4]): R is a solution of $U \circ^* S = T$ iff R^* is a solution of $U \circ S = T$ iff R^* is a solution of $U \circ^* S = T$.

Theorem 4 shows an interesting fact: Let $*$ be globalization. Then R^* is a crisp relation, i.e. $R^*(x, y) = 0$ or $R^*(x, y) = 1$ for any x, y. Therefore, due to Theorem 4, $U \circ^* S = T$ has a solution if and only if the ordinary fuzzy relational equation $U \circ S = T$ has a solution which is a crisp relation. For general $*$, $U \circ^* S = T$ has a solution if and only if the ordinary fuzzy relational equation $U \circ S = T$ has a solution R for which $R(x, y)$ is a fixpoint of $*$. Therefore, solvability of $U \circ^* S = T$ can be regarded as solvability of the ordinary $U \circ S = T$ with an additional constraint imposed by the hedge $*$. The following theorem provides criteria of solvability of $U \circ^* S = T$.

Theorem 5: A fuzzy relational equation $U \circ^* S = T$ has a solution iff $(S \triangleleft T^{-1})^{-1}$ is a solution.

Proof: Check the proof of Theorem 11 in [4] (it is valid even with the weaker hedge). ■

Example 3 ([4]): Let $*$ be globalization and consider Łukasiewicz operations on $L = [0, 1]$. Let S and T be represented by matrices $\begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix}$ and $\begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}$. One can check using Lemma 2 that $U \circ S = T$ has a solution. For example, one such solution is $(S \triangleleft T^{-1})^{-1}$ with the representing matrix being $\begin{pmatrix} 1 & 0.7 \\ 0.9 & 0.5 \end{pmatrix}$, i.e.

$$\begin{pmatrix} 1 & 0.7 \\ 0.9 & 0.5 \end{pmatrix} \circ \begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}.$$

On the other hand, there is no solution of $U \circ^* S = T$, i.e. there is no binary matrix B for which

$$B \circ \begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}.$$

This follows from Theorem 5 by observing that $\big((S \lhd T^{-1})^{-1}\big)^*$ whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not a solution.

Theorem 6: A fuzzy relational equation $U \lhd^* S = T$ has a solution iff $T \lhd S^{-1}$ is a solution.

Proof: Since $U \lhd^* S = U^* \lhd S$, we see that if U is a solution then $U^* \subseteq T \lhd S^{-1}$ (use adjointness and standard manipulation). Due to (4), $U^* \subseteq T \lhd S^{-1}$ is equivalent to $U^* \subseteq (T \lhd S^{-1})^*$. Therefore, $T = U \lhd^* S = U^* \lhd S \supseteq (T \lhd S^{-1})^* \lhd S \supseteq (T \lhd S^{-1}) \lhd S = T$, which shows that $(T \lhd S^{-1}) \lhd^* S = T$. ■

Example 4: Consider Łukasiewicz operations on $L = \{0, 0.25, 0.5, 0.75, 1\}$. Let S and T be represented by matrices

$$\begin{pmatrix} 0.75 & 0.25 & 0 \\ 1 & 0.75 & 1 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0 \end{pmatrix}.$$

Suppose we want to find a solution of $U \lhd^* S = T$ which contains three degrees $K = \{0, 0.5, 1\}$. The hedge $*_2$ from Example 1 satisfies $K = \mathrm{fix}(*_2)$. One can check using Theorem 6 that $U \lhd^* S = T$ has a solution. For example, one such solution is $T \lhd S^{-1}$. Therefore, $(T \lhd S^{-1})^*$ with the representing matrix being

$$\begin{pmatrix} 0.5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

is solution which contains degrees from K.

Theorem 7: A fuzzy relational equation $R \rhd^* U = T$ has a solution iff $R^{-1} \rhd T$ is a solution.

Proof: Since $R \rhd^* U = R \rhd U^*$, we see that if U is a solution then $U^* \subseteq R^{-1} \rhd T$ (use adjointness and standard manipulation). Due to (4), $U^* \subseteq R^{-1} \rhd T$ is equivalent to $U^* \subseteq (R^{-1} \rhd T)^*$. Therefore, $T = R \rhd^* U = R \rhd U^* \supseteq R \rhd (R^{-1} \rhd T)^* \supseteq R \rhd (R^{-1} \rhd T) = T$, which shows that $R \rhd^* (R^{-1} \rhd T) = T$. ■

Example 5: Consider Łukasiewicz operations on $L = \{0, 0.25, 0.5, 0.75, 1\}$. Let R and T be represented by matrices

$$\begin{pmatrix} 0.75 & 0 & 0.75 \\ 0.5 & 0.25 & 0.75 \\ 1 & 0.5 & 1 \end{pmatrix}, \begin{pmatrix} 0.75 & 0.5 \\ 1 & 0.75 \\ 1 & 1 \end{pmatrix}.$$

Suppose we want to find a solution of $R \rhd^* U = T$ which contains four degrees $K = \{0, 0.25, 0.5, 1\}$. The hedge $*_4$ from Example 1 satisfies $K = \mathrm{fix}(*_4)$. One can check using Theorem 7 that $R \rhd^* U = T$ has a solution. For example, one such solution is $R^{-1} \rhd T$. Therefore, $(R^{-1} \rhd T)^*$ with the representing matrix being

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

is solution which contains degrees from K.

Next, we look for solutions of systems of fuzzy relational equations.

In what follows, we need additional property of hedges:

$$\big(\textstyle\bigwedge_{i \in I} a_i\big)^* = \textstyle\bigwedge_{i \in I} a_i^*. \tag{14}$$

Note that (14) is satisfied if, e.g., I is finite and the set of truth degrees is linearly ordered, which is an important practical case.

Theorem 8: System $\mathcal{E} = \{U \circ^* S_j = T_j \mid j \in J\}$ of fuzzy relational equations has a solution iff $\bigcap_{j \in J} (S_j \lhd T_j^{-1})^{-1}$ is a solution of $\mathcal{E}$.

Proof: If U is a solution of $\mathcal{E}$ then $U \circ^* S_j = T_j$ for all $j \in J$. By Theorem 5, $(S_j \lhd T_j^{-1})^{-1}$ is a solution of j-th equation of system $\mathcal{E}$. Moreover, from the proof of Theorem 5 we get that $U^* \subseteq (S_j \lhd T_j^{-1})^{-1}$ for all $j \in J$, therefore $U^* \subseteq \bigcap_{j \in J} (S_j \lhd T_j^{-1})^{-1}$. Due to (4) and (14), $U^* \subseteq \bigcap_{j \in J} (S_j \lhd T_j^{-1})^{-1}$ is equivalent to $U^* \subseteq \bigcap_{j \in J} \big((S_j \lhd T_j^{-1})^{-1}\big)^*$.

Therefore, we have

$$\begin{aligned} T_j = U \circ^* S_j = U^* \circ S_j &\subseteq \textstyle\bigcap_{i \in J} \big((S_i \lhd T_i^{-1})^{-1}\big)^* \circ S_j \\ &\subseteq \textstyle\bigcap_{i \in J} (S_i \lhd T_i^{-1})^{-1} \circ S_j \subseteq (S_j \lhd T_j^{-1})^{-1} \circ S_j = T_j. \end{aligned}$$

So $\bigcap_{i \in J} (S_i \lhd T_i^{-1})^{-1} \circ^* S_j = T_j$ for all $j \in J$. ■

Theorem 9: System $\mathcal{E} = \{U \lhd^* S_j = T_j \mid j \in J\}$ of fuzzy relational equations has a solution iff $\bigcap_{j \in J} (T_j \lhd S_j^{-1})$ is a solution of $\mathcal{E}$.

Proof: If U is a solution of $\mathcal{E}$ then $U \lhd^* S_j = T_j$ for all $j \in J$. By Theorem 6, $T_j \lhd S_j^{-1}$ is a solution of j-th equation of system $\mathcal{E}$. Moreover, from the proof of Theorem 6 we get that $U^* \subseteq T_j \lhd S_j^{-1}$ for all $j \in J$, therefore $U^* \subseteq \bigcap_{j \in J} (T_j \lhd S_j^{-1})$. Due to (4) and (14), $U^* \subseteq \bigcap_{j \in J} (T_j \lhd S_j^{-1})$ is equivalent to $U^* \subseteq \bigcap_{j \in J} (T_j \lhd S_j^{-1})^*$.

Therefore, we have

$$\begin{aligned} T_j = U \lhd^* S_j = U^* \lhd S_j &\supseteq \textstyle\bigcap_{i \in J} (T_i \lhd S_i^{-1})^* \lhd S_j \\ &\supseteq \textstyle\bigcap_{i \in J} (T_i \lhd S_i^{-1}) \lhd S_j \supseteq (T_j \lhd S_j^{-1}) \lhd S_j = T_j. \end{aligned}$$

So $\bigcap_{i \in J} (T_i \lhd S_i^{-1}) \lhd^* S_j = T_j$ for all $j \in J$. ■

Theorem 10: System $\mathcal{E} = \{R_j \rhd^* U = T_j \mid j \in J\}$ of fuzzy relational equations has a solution iff $\bigcap_{j \in J} (R_j^{-1} \rhd T_j)$ is a solution of $\mathcal{E}$.

Proof: Similar to the proof of Theorem 9. ■

V. Constraints Based on Compositions of Fuzzy Relations with Hedges

This section shows that $\circ^*$-compositions of fuzzy relations play an important role in formulating constraints on data with graded attributes. Furthermore, we show that the composition is related to semantics of fuzzy attribute implications.

Consider the following problem as a motivation. We have a collection $\mathcal{S}$ of subsets of Y, i.e. $\mathcal{S} \subseteq 2^Y$. Suppose that $\mathcal{S}$ is a data set resulting as an output of some kind of data analysis, each $M \in \mathcal{S}$ being an "interesting subset of Y"

describing, e.g., an interesting pattern in data. In many cases, we may wish to focus on certain subsets of $\mathcal{S}$ satisfying further requirements, only. For instance, we may formulate a constraint by a binary relation $R \subseteq Y \times Y$, $\langle x, y\rangle \in R$ saying that if x is present in a pattern, y should also be present. We say that $M \subseteq Y$ is compatible with constraint $R \subseteq Y \times Y$ (shortly, M is R-compatible) iff for each $x, y \in Y$, if $x \in M$ and $\langle x, y\rangle \in R$, then $y \in M$. Therefore, M is compatible with constraint R iff $M^R \subseteq M$, where M^R is defined as follows:

$$M^R = \bigl\{y \in Y \,|\, x \in M \text{ and } \langle x, y\rangle \in R\bigr\}. \tag{15}$$

Sets defined by (15) can also be used to compute the least R-compatible superset of M in case M is not compatible with R. Indeed, using (15), we can define a nondecreasing chain of sets $M^R_0 \subseteq M^R_1 \subseteq M^R_2 \subseteq \cdots$ as follows:

$$M^R_i = \begin{cases} M, & \text{for } i = 0, \\ M^R_{i-1} \cup (M^R_{i-1})^R, & \text{for } i \geq 1. \end{cases} \tag{16}$$

Finally, we define $cl_R(M)$ as a union of the previous chain:

$$cl_R(M) = \textstyle\bigcup_{i=0}^{\infty} M^R_i. \tag{17}$$

It can be easily seen that $cl_R : 2^Y \to 2^Y$ is a closure operator. Furthermore, for each M, $cl_R(M)$ is the least R-compatible subset of Y containing M. There is another way to express the same constraint. Namely, we can use simple if-then rules, called attribute implications [14], prescribing the same constraint as the relation R. Attribute implications are formulas of the from $A \Rightarrow B$, where $A, B \subseteq Y$. An attribute implication $A \Rightarrow B$ is true (valid) in $M \subseteq Y$, written $M \models A \Rightarrow B$, if the following condition is satisfied:

$$\text{if } A \subseteq M, \text{ then } B \subseteq M.$$

A subset $M \subseteq Y$ is called a model of a set T of attribute implications if, for each $A \Rightarrow B \in T$, $M \models A \Rightarrow B$. Now, for each binary relation $R \subseteq Y \times Y$ representing a constraint, we can define the following set of attribute implications:

$$T = \bigl\{\{x\} \Rightarrow \{y\} \,|\, \langle x, y\rangle \in R\bigr\}. \tag{18}$$

By definition, $M \models \{x\} \Rightarrow \{y\}$ iff $x \in M$ implies $y \in M$. Therefore, $M \subseteq Y$ is compatible with constraint R iff M is a model of T defined by (18). Also, $cl_R(M)$ is the least model of T containing M.

Hence, there are two equivalent ways to describe simple if-then constraints: via binary relations and particular sets of attribute implications. We now look at the problems of constrains in graded setting and we will see that the $\circ^*$-compositions of fuzzy relations plays an important role here.

First, let Y be a finite nonempty set of attributes. Let $\mathcal{S} \subseteq \mathbf{L}^Y$ be a collection of fuzzy sets in the universe Y; each $M \in \mathcal{S}$ can be seen as an "interesting" fuzzy set in Y. Furthermore, assume that a binary fuzzy relation $R \in \mathbf{L}^{Y \times Y}$ is given. Degrees $R(x, y) \in L$ can be interpreted as degrees to which a dependency "if x is present, then y is present" is prescribed. Therefore, R represents a *graded constraint.* Note that dependencies of this type are by no means artificial. Indeed, on a daily basis, we deal with dependencies the validity of which is, e.g., almost true but not fully true. One example of such dependency may be *"if oil is expensive, prices of imported products are rising"* which may be considered as *more or less true*.

A question is, how one should formulate compatibility in case of graded constraints. One way is outlined in the sequel. Recall that according to (15), $y \in M^R$ iff there is $x \in Y$ such that $x \in M$ and $\langle x, y\rangle \in R$. If we express this existential condition in graded setting, we come up with the following definition:

$$M^R(y) = \textstyle\bigvee_{x \in Y} \bigl(M(x)^* \otimes R(x, y)\bigr), \tag{19}$$

where $M \in \mathbf{L}^Y$ is a fuzzy set, $R \in \mathbf{L}^{Y \times Y}$ is a binary fuzzy relation, and $M^R \in \mathbf{L}^Y$ defined by (19) is a fuzzy set in Y. Described verbally, the degree $M^R(y)$ to which $y \in Y$ belongs to M^R is defined as the degree to which "there is $x \in Y$ such that x belongs to M and x and y are R-related". Since we have used * in (19), we put more emphasis on the fact whether "x belongs to M". If * is globalization, we take into account only those x's which "fully belong to M", i.e. those for which $M(x) = 1$. Using (19) instead of (15), we can define M^R_i ($i \geq 0$) and cl_R in much the same way as in (16) and (17).

We can now say that $M \in \mathbf{L}^Y$ is compatible with constraint $R \in \mathbf{L}^{Y \times Y}$ (shortly, M is R-compatible) iff $M^R \subseteq M$. Clearly, this definition is a graded extension of the previous notion of compatibility. If our structure of truth degrees is the two-valued Boolean algebra, we have exactly the same notion as before. Furthermore, it can be shown that $cl_R(M)$ induced by (19) is the least superset of M which is R-compatible. Note that (19) can be seen as a particular $\circ^*$-composition of fuzzy relations. Namely, instead of $M \in \mathbf{L}^Y$, we can consider a binary fuzzy relation $M' \in \mathbf{L}^{Z \times Y}$, where $Z = \{z\}$, defined by $M'(z, x) = M(x)$ $(x \in Y)$. Then, for each $y \in Y$,

$$\begin{aligned} M^R(y) &= \textstyle\bigvee_{x \in Y} \bigl(M(x)^* \otimes R(x, y)\bigr) \\ &= \textstyle\bigvee_{x \in Y} \bigl(M'(z, x)^* \otimes R(x, y)\bigr) = \bigl(M' \circ^* R\bigr)(z, y). \end{aligned}$$

Hence, up to formalism, M^R results as a $\circ^*$-composition of binary fuzzy relations. One may argue that up till now the presence and role of * in (19) was not properly justified. The presence of * is important because we want to have a reasonable model-theoretical representation of our constrains by fuzzy attribute implications [8] which are graded counterparts of the ordinary attribute implications [14].

Recall that a fuzzy attribute implication (over Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes) [8]. For fuzzy set $M \in \mathbf{L}^Y$ of attributes, we define a degree $||A \Rightarrow B||_M \in L$ to which $A \Rightarrow B$ is true in M by

$$||A \Rightarrow B||_M = S(A, M)^* \to S(B, M). \tag{20}$$

Observe that $||A \Rightarrow B||_M$ is a degree to which it is true that "if A is (very) contained in M, then B is contained in M". Let T be a set of FAIs over Y. $M \in \mathbf{L}^Y$ is called a model of T if $||A \Rightarrow B||_M = 1$ for each $A \Rightarrow B \in T$. In [9], we have shown that the system of all models of T is a particular fuzzy closure system [6]. As a consequence, for each set T of FAIs over Y and a fuzzy set $M \in \mathbf{L}^Y$, there is a uniquely determined least model of T which contains M as its subset. The following assertion shows that cl_R is an operator sending each M to the lest model of certain theory induced by R.

Theorem 11: Let $R \in \mathbf{L}^{Y \times Y}$ be a binary fuzzy relation, $M \in \mathbf{L}^Y$ be a fuzzy set. Then, $cl_R(M)$ is the least model of the following set of FAIs

$$T = \big\{\{{}^1/x\} \Rightarrow \{{}^{R(x,y)}/y\} \,|\, x, y \in Y\big\},$$

such that $M \subseteq cl_R(M)$.

Proof: Let $M \in \mathbf{L}^Y$ be a fuzzy set. For each fuzzy set $M \in \mathbf{L}^Y$, denote by M^T a fuzzy set $M^T \in \mathbf{L}^Y$ defined by

$$M^T = M \cup \bigcup\big\{S(A, M)^* \otimes B \,|\, A \Rightarrow B \in T\big\}.$$

Using results from [9], a fuzzy set $cl_T(M)$ defined by

$$cl_T(M) = M \cup M^T \cup (M^T)^T \cup ((M^T)^T)^T \cup \cdots$$

is the least model of T containing M. Now, observe that

$$\begin{aligned} M^T &= M \cup \bigcup\big\{S(A, M)^* \otimes B \,|\, A \Rightarrow B \in T\big\} \\ &= M \cup \bigcup\big\{S(\{{}^1/x\}, M)^* \otimes \{{}^{R(x,y)}/y\} \,|\, \\ &\qquad\qquad \{{}^1/x\} \Rightarrow \{{}^{R(x,y)}/y\} \in T\big\} \\ &= M \cup \bigcup\big\{S(\{{}^1/x\}, M)^* \otimes \{{}^{R(x,y)}/y\} \,|\, x, y \in Y\big\} \\ &= M \cup \bigcup\big\{M(x)^* \otimes \{{}^{R(x,y)}/y\} \,|\, x, y \in Y\big\} \\ &= M \cup \bigcup\big\{\{{}^{M(x)^* \otimes R(x,y)}/y\} \,|\, x, y \in Y\big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} M^T(y) &= M \cup \bigcup\big\{\{{}^{M(x)^* \otimes R(x,y)}/y\} \,|\, x \in Y\big\}(y) \\ &= M(y) \vee \bigvee\big\{M(x)^* \otimes R(x, y) \,|\, x \in Y\big\} \\ &= M(y) \vee M^R(y), \end{aligned}$$

i.e. $cl_T(M) = cl_R(M)$, proving the claim. ■

As a consequence of the previous assertion and results in [6], [9], we get that the set of all R-compatible fuzzy sets is an $\mathbf{L}^*$-closure system [9] and that cl_R, defined using $\circ^*$-compositions of particular fuzzy relations, is an $\mathbf{L}^*$-closure operator associated with that system. This observation has several benefits. For instance, fixed points of $\mathbf{L}^*$-closure operators can be computed using a polynomial time-delay algorithm. Furthermore, we get that constraints formulated by fuzzy relations belong to a larger family of constraints formulated by fuzzy closure operators studied in [10].

VI. Future Research

Future research will focus mainly on approximate solutions of fuzzy relational equations with hedges. In addition, we plan to use hedge-based compositions for factorization of matrices, i.e. decomposition into two matrices, which contain truth degrees. In particular, we want to extend our recent results on optimal decompositions of such matrices to decompositions constrained, e.g., by the requirement of one matrix be a binary one.

Added when preparing final version. Recently, we found that binary solutions of a particular type of fuzzy relational equations are discussed in [13] (brief comparison: the problem is a particular case of ours, because of the types of fuzzy relational equations and because we consider more general constrained solutions, i.e. not only binary solutions).

Acknowledgment

Supported by grant No. 1ET101370417 of GA AV ČR, by grant No. 201/05/0079 of the Czech Science Foundation, and by institutional support, research plan MSM 6198959214.

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