Reducing attribute implications from data tables with fuzzy attributes to tables with binary attributes

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Abstract

The paper deals with if-then rules from data tables with fuzzy attributes. The main result is a general "reduction theorem" of the form: an if-then rule is true in a data table with fuzzy attributes if and only if a suitably transformed if-then rule is true in a suitably transformed data table with crisp attributes. We show consequences and applications of the reduction theorem plus some auxiliary results.

Keywords: fuzzy logic, if-then rule, attribute implication

1. Introduction and problem setting

Methods for obtaining if-then rules (implications) from data tables describing objects and their attributes are of the most popular ones. The well-known mining of association rules used in data mining is an example.

In our paper, we are interested in if-then rules generated from data with fuzzy attributes: rows and columns of data table correspond to objects $x \in X$ and attributes $y \in Y$, respectively. Table entries $I(x,y)$ are truth degrees to which object x has attribute y. Rules are of the form $A \Rightarrow B$, where A and B are collections of attributes, with the meaning: if an object has all the attributes of A then it has all attributes of B. In crisp setting, attribute implications were thoroughly investigated in databases and data analysis. Good references are [7, 10, 6]. Attribute implications in fuzzy setting were for the first time studied in [11]. Our paper is a continuation of [2, 3, 4]. We provide an answer the following question: Can a data table $\mathcal{T}$ with fuzzy attributes be transformed to a table $\mathcal{T}'$ with crisp attributes in such a way that an attribute implication $A \Rightarrow B$ is true in the original data table $\mathcal{T}$ if and only if a suitable attribute implication $A' \Rightarrow B'$ is true in $\mathcal{T}'$? We show a positive answer in a constructive manner. We present a theorem describing a data transformation procedure from fuzzy to crisp. As an example of application of the theorem, we show how to obtain a complete set of attribute implications from data with fuzzy attributes. As a by-product, we show several formulas related to evaluation of truth degree of if-then rules.

2. Preliminaries

As a set of truth degrees equipped with suitable operations (truth functions of logical connectives) we use a complete residuated lattice with a truth-stressing hedge (shortly, a hedge), i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$; for each $a,b,c \in L$; hedge * satisfies (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a,b \in L$. Elements a of L are called truth degrees (usually, $L \subseteq [0,1]$). $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true". By $\mathbf{L}^U$ (or L^U) we denote the collection of all fuzzy sets in a universe U, i.e. mappings A of U to L. For $A,B \in L^U$, the degree $S(A,B)$ to which A is a subset of B is defined by $S(A,B) = \bigwedge_{u \in U} A(u) \rightarrow B(u)$. In the following we use well-known properties of residuated lattices, hedges, fuzzy sets and fuzzy relations which can be found in [1, 8].

3. Fuzzy attribute implications: reduction results

3.1. Definition and validity

In this section we recall basic notions related to attribute implications. Let X and Y be sets of objects and attributes, respectively, $I \in \mathbf{L}^{X \times Y}$ be a fuzzy relation between X and Y with $I(x,y)$ being interpreted as a degree to which object $x \in X$ has attribute $y \in Y$. The triplet $\langle X, Y, I\rangle$ will be called a data table with fuzzy attributes. A *fuzzy attribute implication* (*in attributes* Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$. The intended meaning of $A \Rightarrow B$ is the following: "if it is (very) true that an object has attributes A, then it has also attributes B".

For an $\mathbf{L}$-set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* $M \in \mathbf{L}^Y$:

$$||A \Rightarrow B||_M = S(A,M)^* \rightarrow S(B,M). \qquad (1)$$

For a system $\mathcal{M} \subseteq \mathbf{L}^Y$ of $\mathbf{L}$-sets in Y we define a degree

$||A \Rightarrow B||_{\mathcal{M}} \in L$ to which $A \Rightarrow B$ holds in $\mathcal{M}$ by $||A \Rightarrow B||_{\mathcal{M}} = \bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M$. Now we can define a *degree* $||A \Rightarrow B||_{\langle X,Y,I\rangle} \in L$ *to which* $A \Rightarrow B$ *holds in (each row of) data table* $\langle X,Y,I\rangle$ by $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{M}}$, where $\mathcal{M} = \{I_x \mid x \in X\}$ with $I_x(y) = I(x,y)$ for each $y \in Y$. Clearly, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is just the truth degree of "for each object $x \in X$: if (it is very true that) x has all attributes from A then x has all attributes from B". To make $*$ explicit, we also write $||A \Rightarrow B||^*_{\mathcal{M}}$ instead of $||A \Rightarrow B||_{\mathcal{M}}$.

A useful structure derived from $\langle X,Y,I\rangle$ which is related to attribute implications is a so-called fuzzy concept lattice [1, 5]. Let $*_X$ and $*_Y$ be hedges (their meaning will become apparent later). For $\mathbf{L}$-sets $A \in \mathbf{L}^X$ ($\mathbf{L}$-set of objects), $B \in \mathbf{L}^Y$ ($\mathbf{L}$-set of attributes) we define $\mathbf{L}$-sets $A^{\uparrow} \in \mathbf{L}^Y$ ($\mathbf{L}$-set of attributes), $B^{\downarrow} \in \mathbf{L}^X$ ($\mathbf{L}$-set of objects) by $A^{\uparrow}(y) = \bigwedge_{x\in X}\big(A(x)^{*_X} \to I(x,y)\big)$, and $B^{\downarrow}(x) = \bigwedge_{y\in Y}\big(B(y)^{*_Y} \to I(x,y)\big)$. We put $\mathcal{B}(X^{*_X},Y^{*_Y},I) = \{\langle A,B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}$. For $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$, put $\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). Operators $^{\downarrow},^{\uparrow}$ form a Galois connection with hedges [5]. $\langle \mathcal{B}(X^{*_X},Y^{*_Y},I), \leq\rangle$ is called a *fuzzy concept lattice* induced by $\langle X,Y,I\rangle$. For $*_Y = \mathrm{id}_L$ (identity), we write only $\mathcal{B}(X^{*_X},Y,I)$. Elements $\langle A,B\rangle$ of $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ are naturally interpreted as concepts (clusters) hidden in the input data represented by I. Namely, $A^{\uparrow} = B$ and $B^{\downarrow} = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. These conditions formalize the definition of a concept as developed in Port-Royal logic; A and B are called the *extent* and the *intent* of the concept $\langle A,B\rangle$, respectively, and represent the collection of all objects and all attributes covered by the particular concept. Furthermore, $\leq$ models the natural subconcept-superconcept hierarchy.

For each $\langle X,Y,I\rangle$ we consider a set $\mathrm{Int}(X^{*_X},Y^{*_Y},I) \subseteq \mathbf{L}^Y$ of all intents of concepts of $\mathcal{B}(X^{*_X},Y^{*_Y},I)$, i.e.

$$\mathrm{Int}(X^{*_X},Y^{*_Y},I) = \\ = \{B \in \mathbf{L}^Y \mid \langle A,B\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I) \text{ for some } A \in \mathbf{L}^X\}.$$

For details, we refer to [1, 2, 5, 3, 4].

3.2. Degree of validity of an implication

This section presents results concerning formulas for $||A \Rightarrow B||_{\langle X,Y,I\rangle}$. First, for a hedge $*$ on $\mathbf{L}$ put $\mathrm{fix}(*) = \{a \in L \mid a^* = a\}$ (fixpoints of $*$). Furthermore, for $\bullet, * : L \to L$ put $\bullet \leq *$ iff $a^\bullet \leq a^*$ for each $a \in L$. Then we have

Lemma 1 *For hedges $*$ and $\bullet$ on a complete residuated lattice $\mathbf{L}$ we have $\bullet \leq *$ iff $\mathrm{fix}(\bullet) \subseteq \mathrm{fix}(*)$.*

Proof. "$\Rightarrow$": If $a \in \mathrm{fix}(\bullet)$ then $a = a^\bullet \leq a^* \leq a$, i.e. $a = a^*$, whence $a \in \mathrm{fix}(*)$. "$\Leftarrow$": Since $a^\bullet \leq a$, we have $a^{\bullet *} \leq a^*$. Since $a^\bullet \in \mathrm{fix}(\bullet)$ and $\mathrm{fix}(\bullet) \subseteq \mathrm{fix}(*)$, we have $a^\bullet = a^{\bullet *}$. This gives $a^\bullet = a^{\bullet *} \leq a^*$. □

Lemma 2 *For $A,B,M \in \mathbf{L}^Y$, and hedges $\bullet$ and $*$ with $\bullet \leq *$ we have $||A \Rightarrow B||_M = \bigwedge_{a\in L} S(a^* \otimes A, M)^\bullet \to S(a^* \otimes B, M) = S(S(A,M)^* \otimes B, M)$.*

Proof. First, using $S(a \otimes A, B) = a \to S(A,B)$ (easy to check), we get $||A \Rightarrow B||_M = S(A,M)^* \to S(B,M) = S(S(A,M)^* \otimes B, M)$.
Second, we check both inequalities of $||A \Rightarrow B||_M = \bigwedge_{a\in L} S(a^* \otimes A,M)^\bullet \to S(a^* \otimes B,M)$. "$\leq$" is true iff for each $a \in L$ we have $S(a^* \otimes A,M)^\bullet \otimes ||A \Rightarrow B||_M \leq S(a^* \otimes B,M)$ and since $S(a^* \otimes B,M) = a^* \to S(B,M)$, the latter inequality is equivalent to $a^* \otimes S(a^* \otimes A,M)^\bullet \otimes ||A \Rightarrow B||_M \leq S(B,M)$ which is true. Indeed, $a^* \otimes S(a^* \otimes A,M)^\bullet \otimes ||A \Rightarrow B||_M \leq a^* \otimes S(a^* \otimes A,M)^* \otimes ||A \Rightarrow B||_M = a^* \otimes (a^* \to S(A,M))^* \otimes ||A \Rightarrow B||_M \leq a^* \otimes (a^* \to S(A,M)^*) \otimes ||A \Rightarrow B||_M \leq S(A,M)^* \otimes (S(A,M)^* \to S(B,M)) \leq S(B,M)$. To check "$\geq$", observe that $\bigwedge_{a\in L} S(a^* \otimes A,M)^\bullet \to S(a^* \otimes B,M) \leq$ (for $a = S(A,M)$) $\leq S(S(A,M)^* \otimes A,M)^\bullet \to S(S(A,M)^* \otimes B,M) = 1^\bullet \to S(S(A,M)^* \otimes B,M) = ||A \Rightarrow B||_M$. □

Theorem 3 (degree of validity of attribute implication) *For a data table $\langle X,Y,I\rangle$ with fuzzy attributes, hedges $\bullet$ and $*$ with $\bullet \leq *$, and an attribute implication $A \Rightarrow B$, the following values are equal:*

$||A \Rightarrow B||_{\langle X,Y,I\rangle}$,
$\bigwedge_{x\in X, a\in L} S(a^* \otimes A, \{{}^1/x\}^{\uparrow})^\bullet \to S(a^* \otimes B, \{{}^1/x\}^{\uparrow})$,
$\bigwedge_{x\in X, a\in L} S(A, \{{}^a/x\}^{\uparrow})^\bullet \to S(B, \{{}^a/x\}^{\uparrow})$,
$\bigwedge_{a\in L} ||a^* \otimes A \Rightarrow a^* \otimes B||_{\langle X,Y,I\rangle}$,
$\bigwedge_{M\in \mathrm{Int}(X^*,Y,I)} S(A,M)^\bullet \to S(B,M)$,
$||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$, $S(B, A^{\downarrow\uparrow})$.

Proof. Due to the limited space we give only a sketch of proof. $||A \Rightarrow B||_{\langle X,Y,I\rangle} = \bigwedge_{x\in X,a\in L} S(a^* \otimes A, \{{}^1/x\}^{\uparrow})^\bullet \to S(a^* \otimes B, \{{}^1/x\}^{\uparrow})$ follows directly from Lemma 2, the definition of $||A \Rightarrow B||_{\langle X,Y,I\rangle}$, and the fact that $\{{}^1/x\}^{\uparrow} = I_x$. $\bigwedge_{x\in X,a\in L} S(a^* \otimes A, \{{}^1/x\}^{\uparrow})^\bullet \to S(a^* \otimes B, \{{}^1/x\}^{\uparrow}) = \bigwedge_{x\in X,a\in L} S(A, \{{}^a/x\}^{\uparrow})^\bullet \to S(B, \{{}^a/x\}^{\uparrow})$ follows from $S(C, \{{}^c/x\}^{\uparrow}) = S(c^* \otimes C, \{{}^1/x\}^{\uparrow})$ which is easy to verify. $\bigwedge_{a\in L} ||a^* \otimes A \Rightarrow a^* \otimes B||_{\langle X,Y,I\rangle}$ equals $\bigwedge_{x\in X,a\in L} S(a^* \otimes A, \{{}^1/x\}^{\uparrow})^\bullet \to S(a^* \otimes B, \{{}^1/x\}^{\uparrow})$ almost directly by definition. $||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} \leq \bigwedge_{M\in\mathrm{Int}(X^*,Y,I)} S(A,M)^\bullet \to S(B,M)$ follows from the fact that for each $M \in \mathrm{Int}(X^*,Y,I)$ we have $S(A,M)^* \to S(B,M) \leq S(A,M)^\bullet \to S(B,M)$ and from $||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = \bigwedge_{M\in\mathrm{Int}(X^*,Y,I)} S(A,M)^* \to S(B,M)$. $\bigwedge_{M\in\mathrm{Int}(X^*,Y,I)} S(A,M)^\bullet \to S(B,M) \leq \bigwedge_{x\in X,a\in L} S(A, \{{}^a/x\}^{\uparrow})^\bullet \to S(B, \{{}^a/x\}^{\uparrow})$ follows since for each $a \in L$ and $x \in X$ we have $\{{}^a/x\}^{\uparrow} \in \mathrm{Int}(X^*,Y,I)$. The remaining relationships are easy or can be found in [3]. □

Remark (1) Note first that taking $\mathbf{L} = \mathbf{2}$ (in which case the only hedges $*$ and $\bullet$ are the identities on $L = \{0,1\}$), i.e. restricting ourselves to the ordinary (crisp) case, all formulas of Theorem 3 reduce to four well-known conditions of validity of an attribute implication in a data table.

(2) Hedge $\bullet$ of the Theorem 3 can range arbitrarily from globalization (i.e. $1^\bullet = 1$ and $a^\bullet = 0$ for $a < 1$), which is the least hedge, up to $*$ (boundary condition Theorem 3). In particular, taking globalization for $\bullet$, the above formulas simplify. Due to lack of space, we only illustrate this fact on $||A \Rightarrow B||_{\langle X,Y,I\rangle} = \bigwedge_{M\in \mathrm{Int}(X^*,Y,I)} S(A,M)^\bullet \to S(B,M)$. For $\bullet$ being globalization, this formula simplifies to $||A \Rightarrow B||_{\langle X,Y,I\rangle} = \bigwedge_{A\subseteq M\in \mathrm{Int}(X^*,Y,I)} S(B,M)$. For instance, compared to $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$, the foregoing formula says that it is sufficient to take the infimum over intents which fully contain A.

(3) Most often, we are interested in implications $A \Rightarrow B$ which are fully true in $\langle X,Y,I\rangle$, i.e. $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$. Conditions equivalent to $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ can be obtained from Theorem 3 using the fact that $a \leq b$ iff $a \to b = 1$ for $a,b \in L$. Just for illustration, again, from $||A \Rightarrow B||_{\langle X,Y,I\rangle} = \bigwedge_{M\in \mathrm{Int}(X^*,Y,I)} S(A,M)^\bullet \to S(B,M)$ we get that $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ iff for each $M \in \mathrm{Int}(X^*,Y,I)$ we have $S(A,M)^\bullet \leq S(B,M)$. If $\bullet$ is globalization, this is the case iff for each $M \in \mathrm{Int}(X^*,Y,I)$ such that $A \subseteq M$ we have $B \subseteq M$.

3.3. Reduction theorem

In this section we give a positive answer to the question of finding a suitable (from the point of view of validity of implications) transformation of a data table $\mathcal{T} = \langle X,Y,I\rangle$ with fuzzy attributes to a table $\mathcal{T}'$ with crisp attributes (see Section 1.). This question is interesting both conceptually (reducing a property in fuzzy setting to a corresponding property in crisp setting), from the computational point of view (there exist several algorithms for attribute implications in crisp setting, see [6, 10]), as well as from the point of view of establishing an interface between two frameworks.

We will use two hedges, $*_X$ and $*_Y$. $*_X$ is used to evaluate truth degrees of attribute implications by (1), i.e. we use $||A \Rightarrow B||_M^{*_X}$. For brevity, we denote $\mathrm{fix}(*_X)$ and $\mathrm{fix}(*_Y)$ by $*_X(L)$ and $*_Y(L)$, respectively. For $C \in \mathbf{L}^X$ and $D \in \mathbf{L}^Y$ we put $\lfloor C \rfloor = \{\langle x,a\rangle \in X \times *_X(L) \mid a \leq C(x)\}$ and $\lfloor D \rfloor = \{\langle y,a\rangle \in Y \times *_Y(L) \mid a \leq D(y)\}$. For $C' \subseteq X \times *_X(L)$ and $D' \subseteq Y \times *_Y(L)$ we define $\lceil C' \rceil \in \mathbf{L}^X$ and $\lceil D' \rceil \in \mathbf{L}^Y$ by $\lceil C' \rceil(x) = \bigvee\{a \mid \langle x,a\rangle \in C'\}$ and $\lceil D' \rceil(y) = \bigvee\{a \mid \langle y,a\rangle \in D'\}$. It can be shown that $\lceil C' \rceil(x) \in *_X(L)$ and $\lceil D' \rceil(y) \in *_Y(L)$ (use $(\bigvee_k a_k^*)^* = \bigvee_k a_k^*$).

Introduce mappings $^\curlywedge : \mathbf{2}^{X\times *_X(L)} \to \mathbf{2}^{Y\times *_Y(L)}$ and $^\curlyvee : \mathbf{2}^{Y\times *_Y(L)} \to \mathbf{2}^{X\times *_X(L)}$ by $A'^\curlywedge = \lfloor \lceil A' \rceil^\uparrow \rfloor$ and $B'^\curlyvee = \lfloor \lceil B' \rceil^\downarrow \rfloor$ for $A' \in \mathbf{2}^{X\times *_X(L)}$ and $B' \in \mathbf{2}^{Y\times *_Y(L)}$.

Given a data table $\mathcal{T} = \langle X,Y,I\rangle$ with fuzzy attributes, introduce a data table $\mathcal{T}'$ with crisp attributes (transformation of $\mathcal{T}$) by $\mathcal{T}' = \langle X \times *_X(L), Y \times *_Y(L), I^\times\rangle$ with a (crisp) relation $I^\times \subseteq (X \times *_X(L)) \times (Y \times *_Y(L))$ defined by $\langle\langle x,a\rangle,\langle y,b\rangle\rangle \in I^\times$ iff $a \otimes b \leq I(x,y)$. In the following, $\mathcal{T}'$ will always denote the table just defined. The next assertion presents important technical results (proof omitted due to lack of space).

Lemma 4 *(1) The pair $\langle ^\curlywedge, ^\curlyvee\rangle$ forms a Galois connection between $X \times *_X(L)$ and $Y \times *_Y(L)$. (2) $\mathcal{T}'$ is just the data table inducing $\langle ^\curlywedge, ^\curlyvee\rangle$. (3) $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to the (ordinary) concept lattice $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$; an isomorphism is given by sending $\langle A,B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ to $\langle \lfloor A \rfloor, \lfloor B \rfloor\rangle \in \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$.*

The following is our basic reduction theorem connecting validity of attribute implications in $\mathcal{T}$ and validity of implications in $\mathcal{T}'$.

Theorem 5 (reduction theorem) *For a data table $\mathcal{T} = \langle X,Y,I\rangle$ with fuzzy attributes, hedges $*_X, *_Y$, and $A', B' \subseteq Y \times *_Y(L)$, $A \in L^Y$, $B \in *_Y(L)^Y$, we have*
(1) $||A \Rightarrow B||_{\mathcal{T}}^{*_X} = 1$ iff $||\lfloor A \rfloor \Rightarrow \lfloor B \rfloor||_{\mathcal{T}'} = 1$, and
(2) $||A' \Rightarrow B'||_{\mathcal{T}'} = 1$ iff $||\lceil A' \rceil \Rightarrow \lceil B' \rceil||_{\mathcal{T}}^{*_X} = 1$ iff $||\lfloor\lceil A' \rceil\rfloor \Rightarrow \lfloor\lceil B' \rceil\rfloor||_{\mathcal{T}'} = 1$.

Proof. Sketch: The proof uses Lemma 4 and Theorem 3, and some further properties of $^\curlywedge$ and $^\curlyvee$. We check only (1): It can be shown that $\lfloor A \rfloor^{\curlyvee\curlywedge} = \lfloor A^{\uparrow\downarrow} \rfloor$. One can thus easily see that $B \subseteq A^{\downarrow\uparrow}$ iff $\lfloor B \rfloor \subseteq \lfloor A \rfloor^{\curlyvee\curlywedge}$. Since $B \subseteq A^{\downarrow\uparrow}$ is equivalent to $||A \Rightarrow B||_{\mathcal{T}}^{*_X} = 1$ (here one uses $B \in *_Y(L)^Y$) and $\lfloor B \rfloor \subseteq \lfloor A \rfloor^{\curlyvee\curlywedge}$ is equivalent to $||\lfloor A \rfloor \Rightarrow \lfloor B \rfloor||_{\mathcal{T}'}^{*_X} = 1$, (1) is established. □

Remark Theorem 5 is rather general and has several corollaries which are of practical interest.
(1) The role of $*_Y$ is to yield smaller $\mathcal{T}'$. Particularly, if A and B are crisp sets (i.e. $A,B \in \{0,1\}^Y$), which is an interesting case form the practical point of view, we can take globalization for $*_Y$ since $*_Y(L) = \{0,1\}$ and so $A,B \in *_Y(L)^Y$. Then we have just $\mathcal{T}' = \langle X \times *_X(L), Y, I^\times\rangle$ since we can replace $Y \times *_Y(L) = Y \times \{0,1\}$ by Y (then $\langle\langle x,a\rangle,y\rangle \in I^\times$ iff $a \leq I(x,y)$). Compared to that, without using $*_Y$, i.e. with $*_Y$ being identity, we would get $\langle X \times *_X(L), Y \times L, I^\times\rangle$ which is larger than $\mathcal{T}'$. In general, if we have A and B restricted to some $*_Y$ (in that $A,B \in *_Y(L)^Y$), we get a smaller $\mathcal{T}'$ than without using $*_Y$ since $|Y \times *_Y(L)| \leq |Y \times L|$.
(2) For unrestricted A,B, we take identity for $*_Y$. So the reduction context is $\mathcal{T}' = \langle X \times *_X(L), Y \times L, I^\times\rangle$. For the two boundary cases of $*_X$ we have: If $*_X$ is globalization, we can take $\mathcal{T}' = \langle X, Y \times L, I^\times\rangle$. If $*_X$ is identity, we have $\mathcal{T}' = \langle X \times L, Y \times L, I^\times\rangle$. Finally, if $*_X$ is globalization and A,B are crisp, we can take $\mathcal{T}' = \langle X,Y,I^\times\rangle$ which is, in

fact, $\langle X,Y,{}^1I\rangle$. Here, ${}^1I$ is the 1-cut of I, i.e. $\langle x,y\rangle \in I$ iff $I(x,y)=1$.

3.4. Complete sets of attribute implications via reduction theorem

In this section, we show that we can use the reduction theorem to obtain a complete set of attribute implications of $\langle X,Y,I\rangle$. We proceed for $*_Y$ being the identity, i.e. we put no restrictions on premises and conclusions in implications.

For a set T of fuzzy attribute implications, put $\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid ||A \Rightarrow B||_M = 1\}$ (models of T). A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ semantically follows from T is defined by $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Mod}(T)}$. T is called complete with respect to $\langle X,Y,I\rangle$ if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$ for each $A \Rightarrow B$. A complete T contains, via semantic entailment, the information about the validity of all implications. If T is complete and no proper subset of T is complete, then T is called a non-redundant basis. T is complete iff $\mathrm{Mod}(T) = \mathrm{Int}(X^*,Y,I)$ [2]. It is known that a particular complete non-redundant basis T is of the form $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ where $\mathcal{P}$ is a so-called system of pseudointents of $\langle X,Y,I\rangle$ for $*$ [2]. While $\mathcal{P}$ is unique and can be efficiently computed for $*$ being globalization (and thus, in particular, in crisp case), the situation is so far not solved for the other hedges $*$. The next result shows how to obtain a complete set T of attribute implications from data table with fuzzy attributes (via the reduction theorem).

Theorem 6 (complete sets of attribute implications) *For a data table $\mathcal{T} = \langle X,Y,I\rangle$ with fuzzy attributes, let T' be a set of attribute implications which is complete w.r.t. $\mathcal{T}' = \langle X \times *_X(L), Y \times L, I^\times\rangle$. Then $T = \{\lceil A'\rceil \Rightarrow \lceil B'\rceil \mid A' \Rightarrow B' \in T'\}$ is a set of (fuzzy) attribute implications which is complete w.r.t. $\mathcal{T}$.*

Proof. Sketch: Due to the above-mentioned criterion, it is sufficient to check $\mathrm{Mod}(T) = \mathrm{Int}(X^{*_X},Y,I)$. Take any $M \in L^Y$. We have $M \in \mathrm{Mod}(T)$ iff for each $A' \Rightarrow B' \in T'$: M is a model of $\lceil A'\rceil \Rightarrow \lceil B'\rceil$ iff (we omit proof) for each $A' \Rightarrow B' \in T'$ and $a \in L$: $\lfloor a \to M\rfloor$ is a model of $\lfloor\lceil A'\rceil\rfloor \Rightarrow \lfloor\lceil B'\rceil\rfloor$ iff (we omit proof) for each $A' \Rightarrow B' \in T'$: $\lfloor M\rfloor$ is a model of $\lfloor\lceil A'\rceil\rfloor \Rightarrow \lfloor\lceil B'\rceil\rfloor$ iff (since $\lfloor M\rfloor \subseteq C'$ iff $\lfloor M\rfloor \subseteq \lfloor\lceil C'\rceil\rfloor$) for each $A' \Rightarrow B' \in T'$: $\lfloor M\rfloor$ is a model of $A' \Rightarrow B'$ iff (since T' is complete and because of the above criterion) $\lfloor M\rfloor \in \mathrm{Int}(X \times *_X(L), Y \times L, I^\times)$ iff (since $\lfloor M\rfloor \in \mathrm{Int}(X \times *_X(L), Y \times L, I^\times)$ iff $M \in \mathrm{Int}(X^{*_X},Y,I)$) $M \in \mathrm{Int}(X^{*_X},Y,I)$, proving the claim. □

Remark (1) Even if T' from Theorem 6 is non-redundant, T need not be (details omitted). As a result, it is still an important problem to look for direct ways to obtain systems of pseudointents of $\langle X,Y,I\rangle$ for general $*_X$.

(2) A particular T' for $\mathcal{T}'$ which can be efficiently computed is the above-described $T' = \{P' \Rightarrow P'^{\downarrow\uparrow} \mid P' \in \mathcal{P}'\}$ with $\mathcal{P}'$ being the set of all pseudointents of $\mathcal{T}'$ [6].

4. Conclusions and future research

We showed a general reduction result for implications from data with fuzzy attributes, its consequences, and applications. The presented results improve and further develop some previously published results (we will comment on the relationships in a forthcoming paper). Future research will be directed mainly to the study of complete and non-redundant bases of implications from data with fuzzy attributes (cf. the last Remark).

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