Similarity issues in attribute implications from data with fuzzy attributes

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***Abstract*— We study similarity in formal concept analysis of data tables with fuzzy attributes. We focus on similarity related to attribute implications, i.e. rules $A \Rightarrow B$ describing dependencies "each object which has all attributes from A has also all attributes from B". We present several formulas for estimation of similarity of outputs in terms of similarity of inputs. The results answer some natural questions such as how much do truth degrees of $A_1 \Rightarrow B$ and $A_2 \Rightarrow B$ differ in terms of similarity of A_1 to A_2?**

I. Introduction and problem setting

Formal concept analysis (FCA) is a method for analysis of tabular data describing objects and their attributes [16], [17]. Two basic outputs of FCA are concept lattices and attribute implications. A concept lattice is a set of all clusters (called formal concepts) extracted from data, hierarchically ordered by subconcept-superconcept relation. Attribute implications are particular expressions describing certain attribute dependencies. Efficient algorithms are known to compute a concept lattice and a non-redundant set of attribute implications which entail all attribute implications true in data.

In the basic form, attributes are assumed to be bivalent, i.e. either a given attribute y applies to a given object x (indicated by 1 in the data table) or not (indicated by 0). More often than not, attributes are fuzzy rather than bivalent, i.e. an attribute y applies to an object x to a certain degree. FCA of data tables with fuzzy attributes was studied by several authors, we refer to [15] for the first approach, and to [23] and [1]–[11] for the approach we are using in the present paper.

The present paper is a continuation of [1], [5] and [14]. We present results related to similarity in FCA of data with fuzzy attributes. Particularly, we concentrate on similarity issues in fuzzy attribute implications. Our study is motivated by the following questions: Do similar input data lead to similar outputs of FCA attribute implications? Can we obtain estimations of the similarities in question? Can we utilize the similarities to reduce the amount of (input or output) data by putting together similar pieces of data? A study of these problems also tells us about a sensitivity of FCA to exact degrees (in the input data, in the attribute implications) which is an important issue in fuzzy modeling by itself.

In [1], [5], we studied several issues related to similarity in FCA including a computationally efficient method of parameterized factorization of concept lattices by similarity. In [14], we presented results on similarity in a parameterized method of FCA of data with fuzzy attributes with so-called hedges serving as parameters. The above-mentioned results concern formal concepts and concept lattices, i.e. the first output of FCA. The main aim of the present paper is to extend results on similarity to attribute implications, i.e. the second output of FCA.

II. Preliminaries

We use sets of truth degrees equipped with operations (logical connectives) which form complete residuated lattices, i.e. algebras $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, for each $a, b, c \in L$. A truth-stressing hedge (shortly, a hedge) [20], [21] on $\mathbf{L}$ is a unary operation ${}^* : L \rightarrow L$ satisfying (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true" and properties (i)–(iv) have natural interpretations, see [20], [21].

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel: ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = \frac{b}{a}$ else). In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \dots, a_n = 1\} \subseteq [0, 1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization [24]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that a special case of a complete residuated lattice with a hedge is a two-element Boolean algebra of classical (bivalent) logic.

Having $\mathbf{L}$, we define usual notions [2], [18], [20]: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A \colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u \in U} \bigl(A(u) \rightarrow B(u)\bigr), \tag{2}$$

I	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$	
x	$\cdots$	$I(x,y)$	$\cdots$
$\vdots$		$\vdots$	

Fig. 1. Data table with fuzzy attributes

which generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary **L**-relation on $\mathbf{L}^U$). Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Given $A, B \in \mathbf{L}^U$, we define an equality degree

$$A \approx B = \textstyle\bigwedge_{u\in U} \big(A(u) \leftrightarrow B(u)\big), \tag{3}$$

where $\leftrightarrow$ is defined by $(a \to b) \wedge (b \to a)$. It is easily seen that $A \approx B = S(A,B) \wedge S(B,A)$.

A fuzzy relation E in U is called reflexive if for each $u \in U$ we have $E(u,u) = 1$; symmetric if for each $u, v \in U$ we have $E(u,v) = E(v,u)$; transitive if for each $u, v, w \in U$ we have $E(u,v) \otimes E(v,w) \leq E(u,w)$. A fuzzy equivalence [25] in U is a fuzzy relation in U which is reflexive, symmetric, and transitive; a fuzzy equivalence E in U for which $E(u,v) = 1$ implies $u = v$ is called a fuzzy equality. We often denote a fuzzy equivalence by $\approx$ and use an infix notation, i.e. we write $(u \approx v)$ instead of $\approx(u,v)$. If a set U is equipped with a fuzzy equality $\approx$ in U, a fuzzy relation $\preceq$ in U is called a fuzzy order in $\langle U, \approx\rangle$ [2], [25] if $\preceq$ is reflexive, transitive, and antisymmetric w.r.t. $\approx$, i.e. for each $u, v \in U$ we have $(u \preceq v) \wedge (v \preceq u) \leq (u \approx v)$, and if $\preceq$ is compatible with $\approx$, i.e. $(u_1 \preceq v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq (u_2 \preceq v_2)$, see [2] for details.

III. Similarity in fuzzy attribute implications

A. Basic definitions

In this section, we recall basic notions on fuzzy attribute implications and related issues. For details, we refer to [2], [4], [8], [9], [10], [11], [12] and [17], [19], [22].

Let X and Y be sets of objects and attributes, respectively, $I \in \mathbf{L}^{X\times Y}$ be a fuzzy relation between X and Y with $I(x,y)$ being interpreted as a degree to which object $x \in X$ has attribute $y \in Y$. The triplet $\langle X, Y, I\rangle$ is called a data table with fuzzy attributes, see Fig. 1.

Recall that a fuzzy attribute implication (over attributes Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). The intended meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has also all attributes from B". For an **L**-set $M \in \mathbf{L}^Y$ of attributes, we define a degree $||A \Rightarrow B||_M \in L$ to which $A \Rightarrow B$ is valid (true) in M:

$$||A \Rightarrow B||_M = S(A,M)^* \to S(B,M). \tag{4}$$

Here, * is a hedge controlling the meaning of $A \Rightarrow B$. If M is the fuzzy set of all attributes of an object x, then $||A \Rightarrow B||_M$ is the truth degree to which $A \Rightarrow B$ holds for x. We write $||A \Rightarrow B||^*_M$ if we want to make * apparent. For a set $\mathcal{M} \subseteq \mathbf{L}^Y$ (i.e. $\mathcal{M}$ is an ordinary set of fuzzy sets of attributes) we define a degree $||A \Rightarrow B||_{\mathcal{M}} \in L$ to which $A \Rightarrow B$ holds in $\mathcal{M}$ by

$$||A \Rightarrow B||_{\mathcal{M}} = \textstyle\bigwedge_{M\in\mathcal{M}} ||A \Rightarrow B||_M. \tag{5}$$

This enables us to define a validity degree of fuzzy attribute implications in a data table $\langle X, Y, I\rangle$ with fuzzy attributes. Denote by $I_x \in \mathbf{L}^Y$ ($x \in X$) an **L**-set of attributes such that $I_x(y) = I(x,y)$ for each $y \in Y$. Described verbally, I_x is the **L**-set of all attributes of $x \in X$, i.e. in $\langle X, Y, I\rangle$, I_x corresponds to a row labeled x. A degree $||A \Rightarrow B||_{\langle X,Y,I\rangle} \in L$ to which $A \Rightarrow B$ holds in (each row of) $\langle X, Y, I\rangle$ is defined by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{M}}, \tag{6}$$

where $\mathcal{M} = \{I_x \,|\, x \in X\}$.

Note that fuzzy attribute implications are closely related to fuzzy concept lattices. The basic notions follow. Let $*_X$ and $*_Y$ be hedges. For **L**-sets $A \in \mathbf{L}^X$ (**L**-set of objects), $B \in \mathbf{L}^Y$ (**L**-set of attributes) we define **L**-sets $A^\uparrow \in \mathbf{L}^Y$ (**L**-set of attributes), $B^\downarrow \in \mathbf{L}^X$ (**L**-set of objects) by $A^\uparrow(y) = \bigwedge_{x\in X}\big(A(x)^{*_X} \to I(x,y)\big)$, and $B^\downarrow(x) = \bigwedge_{y\in Y}\big(B(y)^{*_Y} \to I(x,y)\big)$. We put $\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A,B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \,|\, A^\uparrow = B, B^\downarrow = A\}$. For $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put $\langle A_1,B_1\rangle \leq \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). Operators $^\downarrow, ^\uparrow$ form a Galois connection with hedges [6]. $\langle \mathcal{B}(X^{*_X}, Y^{*_Y}, I), \leq\rangle$ is called a (fuzzy) concept lattice with hedges $*_X$ and $*_Y$ induced by $\langle X, Y, I\rangle$ [8]. For $*_Y = \mathrm{id}_L$ (identity), we write only $\mathcal{B}(X^{*_X}, Y, I)$. Elements $\langle A,B\rangle$ of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ are naturally interpreted as concepts (clusters) hidden in the input data represented by I. Namely, $A^\uparrow = B$ and $B^\downarrow = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. These conditions formalize the definition of a concept as developed in Port-Royal logic; A and B are called the extent and the intent of the concept $\langle A,B\rangle$, respectively, and represent the collection of all objects and all attributes covered by $\langle A,B\rangle$. $\leq$ models a subconcept-superconcept hierarchy.

For each $\langle X, Y, I\rangle$ we consider a set $\mathrm{Ext}(X^{*_X}, Y^{*_Y}, I) \subseteq \mathbf{L}^X$ of all extents and a set $\mathrm{Int}(X^{*_X}, Y^{*_Y}, I) \subseteq \mathbf{L}^Y$ of all intents of concepts of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, i.e.

$$\mathrm{Ext}(X^{*_X}, Y^{*_Y}, I) = \\ = \{A \in \mathbf{L}^X \mid \langle A,B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I) \text{ for some } B \in \mathbf{L}^Y\},$$

$$\mathrm{Int}(X^{*_X}, Y^{*_Y}, I) = \\ = \{B \in \mathbf{L}^Y \mid \langle A,B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I) \text{ for some } A \in \mathbf{L}^X\}.$$

We write $\mathrm{Int}(X^{*_X}, Y, I) =$ if $*_Y$ is identity, and the like.

B. Similarity and fuzzy attribute implications

This section presents selected results on similarity in attribute implications. We start by the following definition which captures the concept of similarity between two hedges.

Definition 1 ([14]): For hedges $*_1$ and $*_2$ on **L** put

$$(*_1 \preceq *_2) = \bigwedge_{a\in L} (a^{*_1} \to a^{*_2}),$$

$$(*_1 \approx *_2) = \bigwedge_{a\in L} (a^{*_1} \leftrightarrow a^{*_2}).$$

Remark 2: (1) Since $a \leftrightarrow b$ (degree of equivalence of a and b) can be seen as a degree to which degrees a and b are similar, $*_1 \approx *_2$ can be interpreted as a degree to which hedges $*_1$ and $*_2$ are similar (yield similar results). More precisely, using semantics of predicate fuzzy logic, one can see that $*_1 \approx *_2$ is a truth degree of "for each $a \in L$: the result of $*_1$ applied to a is similar to the result of $*_2$ applied to a". Analogously, $*_1 \preceq *_2$ can be interpreted as a degree to which $*_1$ is stronger than $*_2$.

(2) Note that $(*_1 \approx *_2) = (*_1 \preceq *_2) \wedge (*_2 \preceq *_1)$.

The following lemma shows that $\approx$ is indeed a fuzzy equality relation and that $\preceq$ is reflexive and transitive.

Lemma 3 ([14]): $\approx$ is a fuzzy equality relation on the set of all truth stressers on $\mathbf{L}$; $\preceq$ is a fuzzy order on the set of all truth stressers on $\mathbf{L}$ equipped with $\approx$. ■

A degree $||A \Rightarrow B||^*_M$ to which a given fuzzy attribute implication $A \Rightarrow B$ is true in M (again, think of M a a fuzzy set of attributes of a given object) depends on $*$. The reasons for parameterizing by $*$ are discussed elsewhere (e.g. [13]). For the two boundary choices of $*$, namely globalization and identity (see above), we have the following intuitively appealing conditions for $A \Rightarrow B$ being fully true (i.e., true in degree 1) in M:

$$A \subseteq M \text{ implies } B \subseteq M$$

for globalization (i.e., if A is fully contained in M then B is fully contained in M);

$$S(A,M) \leq S(B,M)$$

for identity (i.e., a degree to which A is contained in M is less than or equal to the degree to which B is contained in M). There are, of course other choices of $*$ than globalization and identity and all of them have the same verbal description (the same core meaning). Therefore, a question arises of what is the relationship of the truth degree of $A \Rightarrow B$ when using $*_1$ to the truth degree of $A \Rightarrow B$ when using $*_2$ in terms of similarity $\approx$ of $*_1$ and $*_2$. The next theorem provides an answer.

The first theorem shows how validity of $A \Rightarrow B$ changes if we change *.

Theorem 4: For $M \in \mathbf{L}^M$ and $\mathcal{M} \subseteq \mathbf{L}^M$ we have

$$(*_1 \preceq *_2) \quad \leq \quad ||A \Rightarrow B||^{*_2}_M \rightarrow ||A \Rightarrow B||^{*_1}_M,$$
$$(*_1 \approx *_2) \quad \leq \quad ||A \Rightarrow B||^{*_1}_M \leftrightarrow ||A \Rightarrow B||^{*_2}_M.$$

Proof: The first inequality is true iff

$$(*_1 \preceq *_2) \otimes ||A \Rightarrow B||^{*_2}_M \leq ||A \Rightarrow B||^{*_1}_M$$

which is equivalent to

$$S(A,M)^{*_1} \otimes (*_1 \preceq *_2) \otimes ||A \Rightarrow B||^{*_2}_M \leq S(B,M)$$

which is true (details postponed to a full version). The rest can be proved using the first inequality. ■

Remark 5: In words, the previous theorem says that if $A \Rightarrow B$ is true using ^{*_2} and if ^{*_1} is stronger than ^{*_2} then $A \Rightarrow B$ is true using ^{*_1}, and that if ^{*_1} and ^{*_2} are similar then the degrees to which $A \Rightarrow B$ is true using ^{*_1} and using ^{*_2} are similar.

The next issue is the following. Consider a fixed hedge $*$. To what extent is the truth degree of $A \Rightarrow B$ in a given M dependent on the truth degrees involved in A, B? That is, how is the closeness (similarity) of $||A_1 \Rightarrow B_1||_M$ and $||A_2 \Rightarrow B_2||_M$ dependent of closeness (similarity) of A_1 to A_2 and B_1 to B_2? And, analogously, to what extent is the truth degree of a given $A \Rightarrow B$ in M dependent on the truth degrees involved in M? That is, how is the closeness (similarity) of $||A \Rightarrow B||_{M_1}$ and $||A \Rightarrow B||_{M_2}$ dependent of closeness (similarity) of M_1 to M_2? An answer is provided by the following theorem.

Theorem 6: For a fixed * we have

$$S(A_1,A_2)^* \otimes S(B_2,B_1) \otimes ||A_1 \Rightarrow B_1||_M \leq ||A_2 \Rightarrow B_2||_M,$$
$$S(M_2,M_1)^* \otimes S(M_1,M_2) \otimes ||A \Rightarrow B||_{M_1} \leq ||A \Rightarrow B||_{M_2}.$$

Proof: Using basic properties of complete residuated lattices and hedges we can see that

$$S(A_1,A_2)^* \otimes S(B_2,B_1) \otimes ||A_1 \Rightarrow B_1||_M \leq ||A_2 \Rightarrow B_2||_M$$

is equivalent to

$$S(A_1,A_2)^* \otimes S(A_2,M)^* \otimes S(B_2,B_1) \otimes ||A_1 \Rightarrow B_1||_M \leq S(B_2,M)$$

which is true since

$$\begin{aligned} & S(A_1,A_2)^* \otimes S(A_2,M)^* \otimes S(B_2,B_1) \otimes ||A_1 \Rightarrow B_1||_M \leq \\ \leq \quad & S(A_1,M)^* \otimes S(B_2,B_1) \otimes (S(A_1,M)^* \rightarrow S(B_1,M)) \leq \\ \leq \quad & S(B_2,B_1) \otimes S(B_1,M) \leq S(B_2,M), \end{aligned}$$

proving the first inequality. The rest can be proved analogously (we omit details due to limited scope of the paper). ■

The previous results can be combined to show how validity of a fuzzy attribute implication changes if we change A, B, M, and * simultaneously.

Corollary 7: For a fixed * we have

$$\begin{aligned} & S(A_1,A_2)^* \otimes S(B_2,B_1) \otimes \\ & \otimes S(M_2,M_1)^* \otimes S(M_1,M_2) \otimes ||A_1 \Rightarrow B_1||_{M_1} \leq \\ & \leq ||A_2 \Rightarrow B_2||_{M_1}. \end{aligned}$$ ■

The results can also be extended to validity in systems $\mathcal{M}$ of fuzzy sets of attributes (in particular, validity in a data table). We limit ourselves to the following case. Let $\langle X,Y,I_1\rangle$ and $\langle X,Y,I_2\rangle$ be two tables with fuzzy attributes. Consider degrees $S(I_1,I_2)$ and $I_1 \approx I_2$, i.e.

$$S(I_1,I_2) = \bigwedge_{\langle x,y\rangle \in X \times Y} (I_1(x,y) \rightarrow I_2(x,y))$$

and

$$I_1 \approx I_2 = \bigwedge_{\langle x,y\rangle \in X \times Y} (I_1(x,y) \leftrightarrow I_2(x,y)).$$

$S(I_1,I_2)$ can be interpreted as a degree to which I_1 is included in I_2, i.e. a degree to which entries of table $\langle X,Y,I_1\rangle$ are less than or equal to the corresponding entries of table $\langle X,Y,I_2\rangle$. Analogously, $I_1 \approx I_2$ can be interpreted as a degree to which I_1 is equal to I_2, i.e. a degree to which entries of table $\langle X,Y,I_1\rangle$ are equal (similar, close) to the corresponding entries of table $\langle X,Y,I_2\rangle$. Th following theorem provides us with a basic answer to a question of relationship between degrees of truth of a given fuzzy attribute implication $A \Rightarrow B$ in two different, but possibly similar, tables $\langle X,Y,I_1\rangle$ and $\langle X,Y,I_2\rangle$.

Theorem 8: For a fixed * we have

$$S(I_2,I_1)^* \otimes S(I_1,I_2) \otimes ||A \Rightarrow B||_{I_1} \quad \leq \quad ||A \Rightarrow B||_{I_2},$$
$$(I_2 \approx I_1)^* \otimes (I_1 \approx I_2) \otimes ||A \Rightarrow B||_{I_1} \quad \leq \quad ||A \Rightarrow B||_{I_2}.$$

Proof: Follows from Theorem 6, and from $(I_1 \approx I_2) \leq S(I_1,I_2)$ and $(I_1 \approx I_2) \leq S(I_2,I_1)$. We omit details. ■

We conclude by the following result. Recall first that $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is defined as a degree to which $A \Rightarrow B$ is valid in $\{I_x \mid x \in X\}$ (rows of data table). In bivalent setting, the set $\{I_x \mid x \in X\}$ is, in fact, the set of all intents of all object concepts, see [17]. The set of all object concepts has some important properties. For instance, it is supremally dense, i.e. each formal concept is a supremum of some object concepts. If fuzzy setting, however, $\{I_x \mid x \in X\}$ need not be supremally dense (in $\mathcal{B}(X^*,Y,I)$ which is the corresponding concept lattice, see [8] and [9]). Namely, a supremally dense set of object concepts in fuzzy setting is the set $\mathcal{OB} = \{\langle\{^a/x\}^{\uparrow\downarrow},\{^a/x\}^{\uparrow}\rangle \mid a \in L, x \in X\}$. A problem therefore arises as to what is the relationship of $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to $||A \Rightarrow B||_{\mathcal{OB}}$. The following theorem provides a general answer: It gives a sufficient condition for a subset $\mathcal{M} \subseteq \mathrm{Int}(X^*,Y,I)$ to satisfy $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{M}}$.

Theorem 9: Let $\mathcal{M} \subseteq \mathrm{Int}(X^*,Y,I)$. If for each $x \in X$ there is $\mathcal{M}' \subseteq \mathcal{M}$ such that $I_x = \bigcap \mathcal{M}'$ then

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{M}}.$$

Proof: Sketch: By definitions using the following facts: $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$, see [9]; $\mathcal{M}_1 \subseteq \mathcal{M}_2$ implies $||A \Rightarrow B||_{\mathcal{M}_2} \leq ||A \Rightarrow B||_{\mathcal{M}_1}$; $\bigwedge_{M\in\mathcal{M}} ||A \Rightarrow B||_M \leq ||A \Rightarrow B||_{\bigcap\mathcal{M}}$. ■

Now, since $\mathcal{OB}$ satisfies the conditions of Theorem 9, we have

Corollary 10: $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{OB}}$. ■

IV. Future Research

Future research as well as topics which did not fit the limited extent of this paper include the following:

- factorization of collections of attribute implications by putting together similar implications;
- further results concerning validity of attribute implications;
- results concerning similarity of theories consisting of attribute implications (do similar data tables have similar non-redundant bases of attribute implications? etc.);
- examples demonstrating similarity-estimation results;
- similarity results based on other measures of similarity of fuzzy sets.

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References

[1] Belohlavek R.: Similarity relations in concept lattices. *J. Logic Comput.* 10(6):823–845, 2000.
[2] Belohlavek R.: *Fuzzy Relational Systems: Foundations and Principles*. Kluwer, Academic/Plenum Publishers, New York, 2002.
[3] Belohlavek R.: Concept lattices and order in fuzzy logic. *Ann. Pure Appl. Logic* **128**(2004), 277–298.
[4] Belohlavek R., Chlupová M., Vychodil V.: Implications from data with fuzzy attributes. AISTA 2004 in Cooperation with the IEEE Computer Society Proceedings, 2004, 5 pages, ISBN 2–9599776–8–8.
[5] Belohlavek R., Dvořák J., Outrata J.: Fast factorization by similarity in formal concept analysis of data with fuzzy attributes (submitted). Preliminary version in Proc. CLA 2004, pp. 47–57.
[6] Belohlavek R., Funioková T., Vychodil V.: Galois connections with hedges. In: Yingming Liu, Guoqing Chen, Mingsheng Ying (Eds.): *Fuzzy Logic, Soft Computing & Computational Intelligence: Eleventh International Fuzzy Systems Association World Congress* (Vol. II), 2005, pp. 1250–1255. Tsinghua University Press and Springer, ISBN 7–302–11377–7.
[7] Belohlavek R., Vychodil V.: Implications from data with fuzzy attributes vs. scaled binary attributes. In: FUZZ-IEEE 2005, The IEEE International Conference on Fuzzy Systems, May 22–25, 2005, Reno (Nevada, USA), pp. 1050–1055 (proceedings on CD), abstract in printed proceedings, p. 53, ISBN 0–7803–9158–6.
[8] Belohlavek R., Vychodil V.: Reducing the size of fuzzy concept lattices by hedges. In: FUZZ-IEEE 2005, The IEEE International Conference on Fuzzy Systems, May 22–25, 2005, Reno (Nevada, USA), pp. 663–668 (proceedings on CD), abstract in printed proceedings, p. 44, ISBN 0–7803–9158–6.
[9] Belohlavek R., Vychodil V.: Fuzzy attribute logic: attribute implications, their validity, entailment, and non-redundant basis. In: Yingming Liu, Guoqing Chen, Mingsheng Ying (Eds.): *Fuzzy Logic, Soft Computing & Computational Intelligence: Eleventh International Fuzzy Systems Association World Congress* (Vol. I), 2005, pp. 622–627. Tsinghua University Press and Springer, ISBN 7–302–11377–7.
[10] Belohlavek R., Vychodil V.: Fuzzy attribute logic: syntactic entailment and completeness. *Proceedings of the 8th Joint Conference on Information Sciences*, 2005, pp. 78–81, ISBN 0–9707890–3–3.
[11] Belohlavek R., Vychodil V.: Functional dependencies of data tables over domains with similarity relations. In: Prasad B. (Ed.): IICAI 2005, Proc. 2nd Indian International Conference on Artificial Intelligence, Pune, India, Dec 20–22, 2005, ISBN 0–9727412–1–6, pp. 2486–2504 .
[12] Belohlavek R., Vychodil V.: Axiomatizations of fuzzy attribute logic. In: Prasad B. (Ed.): IICAI 2005, Proc. 2nd Indian International Conference on Artificial Intelligence, Pune, India, Dec 20–22, 2005, ISBN 0–9727412–1–6, pp. 2178–2193
[13] Belohlavek R., Vychodil V.: Attribute implications in a fuzzy setting. In: Missaoui R., Schmid J. (Eds.): ICFCA 2006, LNAI **3874**, pp. 45–60, 2006.
[14] Belohlavek R., Vychodil V.: Estimations of similarity in formal concept analysis of data with graded attributes. Proc. ATDM 2006, Algorithmic Techniques for Data Mining, June 5–7, 2006, Beer-Sheva, Israel, to appear by Springer (series Studies in Computational Intelligence).
[15] Burusco A., Fuentes-Gonzáles R.: The study of the L-fuzzy concept lattice. *Mathware & Soft Computing*, **3**(1994), 209–218.
[16] Carpineto C., Romano G.: *Concept Data Analysis. Theory and Applications.* J. Wiley, 2004.
[17] Ganter B., Wille R.: *Formal Concept Analysis. Mathematical Foundations.* Springer, Berlin, 1999.
[18] Goguen J. A.: L-fuzzy sets. *J. Math. Anal. Appl.* **18**(1967), 145–174.
[19] Guigues J.-L., Duquenne V.: Familles minimales d'implications informatives resultant d'un tableau de données binaires. *Math. Sci. Humaines* **95**(1986), 5–18.
[20] Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.
[21] Hájek P.: On very true. *Fuzzy Sets and Systems* **124**(2001), 329–333.
[22] Maier D.: *The Theory of Relational Databases.* Computer Science Press, Rockville, 1983.
[23] Pollandt S.: *Fuzzy Begriffe.* Springer-Verlag, Berlin/Heidelberg, 1997.
[24] Takeuti G., Titani S.: Globalization of intuitionistic set theory. *Annals of Pure and Applied Logic* **33**(1987), 195–211.
[25] Zadeh L. A.: Similarity relations and fuzzy orderings. *Information Sciences* **3**(1971), 159–176.