Counting Finite Residuated Lattices*

Radim Belohlavek[1,2] and Vilem Vychodil[2]

[1] Dept. Systems Science and Industrial Engineering, Binghamton University—SUNY
Binghamton, NY 13902, USA
rbelohla@binghamton.edu
[2] Dept. Computer Science, Palacky University, Olomouc
Tomkova 40, CZ-779 00 Olomouc, Czech Republic
vilem.vychodil@upol.cz

Abstract. We study finite residuated lattices with up to 11 elements. We present an algorithm for generating all non-isomorphic finite residuated lattices with a given number of elements. Furthermore, we analyze selected properties of all the lattices generated by our algorithm and present summarizing statistics.

1 Introduction and Preliminaries

Problem Setting. Residuated lattices and particular subclasses of residuated lattices distinguished by identities play a key role in fuzzy logic in both the wide and narrow sense [2,4,6,7,9,12,14,16]. In particular, residuated lattices are considered as structures of truth degrees, i.e., scales of truth degrees equipped with truth functions of (fuzzy) logical connectives. Most common choice of a residuated lattice is a structure defined on the real unit interval $[0, 1]$ with multiplication given by a left-continuous t-norm, see [4,9].

In general, non-comparable truth degrees might be of interest for both theoretical and practical reasons (we omit details) which leads us beyond the scope of linearly ordered residuated lattices. Moreover, "small" lattices play an important role. Namely, according to Miller's 7±2 phenomenon well known from psychology [15], humans are able to assign degrees in a consistent manner provided the scale of degrees contains up to 7±2 elements. With more than 7±2 elements, the assignments become inconsistent. Another argument supporting the importance of finite residuated lattices comes from computational considerations. While using $[0, 1]$ is satisfactory in many cases, quite a lot of problems leads to infinite structures if $[0, 1]$ is used (consider just the simple fact that the set of all fuzzy sets in a finite universe is uncountable when $[0, 1]$ is used as a set of truth degrees). Quite often, a natural solution, which is computationally tractable, consists in considering a finite residuated lattice.

These facts bring us to finite residuated lattices with a reasonably small number of elements (7±2, perhaps a bit more). Surprisingly, little has been done in

* Supported by grant No. 1ET101370417 of GA AV ČR, by grant No. 201/05/0079 of the Czech Science Foundation, and by institutional support, research plan MSM 6198959214.

P. Melin et al. (Eds.): IFSA 2007, LNAI 4529, pp. 461–470, 2007.

a systematic study in this field. An interesting exception is [1] which studies t-norms on finite chains. This paper is a continuation of [3] where we investigated properties of all non-isomorphic finite lattices up to eleven elements. In this paper, we focus on fundamental questions related to finite *residuated* lattices. For instance, how many non-isomorphic residuated lattices (MTL-algebras, BL-algebras, ...) with n elements are there? Can we generate the structures in an efficient way? Which properties of finite residuated lattices are frequent for small lattices? Our paper answers several questions of this type.

Preliminaries. Recall that a residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$ (so-called adjointness). More details about residuated lattices and their role in fuzzy logic can be found in [2,6,7,9].

2 Generation of Finite Residuated Lattices

In this section we briefly describe a way to generate residuated lattices of a given size. Given a finite lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$, we describe an algorithm which generates all pairs $\langle\otimes, \rightarrow\rangle$ of adjoint operations on $\mathbf{L}$. Thus, in order to generate all n-element residuated lattices, it suffices to generate all n-element lattices and then to generate all the adjoint pairs. Efficient algorithms for generating finite lattices are available, see [3,11].

Let $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a finite lattice. By definition of a residuated lattice, we are looking for all couples $\langle\otimes, \rightarrow\rangle$ of operations such that $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy adjointness. Note that not all finite lattices admit adjoint operations $\otimes$ and $\rightarrow$. The algorithm which will be described later in this section generates only $\otimes$ (multiplication, truth function of "fuzzy conjunction") and tests a condition which is equivalent to the existence of $\rightarrow$ (residuum, truth function of "fuzzy implication") satisfying adjointness with $\otimes$. Namely, we will take advantage of the following assertion:

Theorem 1. *Let $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a finite lattice, $\langle L, \otimes, 1\rangle$ be a commutative monoid such that $\otimes$ is monotone w.r.t. $\leq$. The the following are equivalent:*

(i) *there exists (unique) $\rightarrow$ satisfying adjointness w.r.t. $\otimes$;*
(ii) *for each $a, b, c \in L$: $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$;*
(iii) *$\rightarrow$ given by $a \rightarrow b = \bigvee\{c \in L \mid a \otimes c \leq b\}$ satisfies adjointness w.r.t. $\otimes$.*

Proof. Follows from finiteness of L and properties of residuated lattices [2]. We omit the proof due to the limited scope of this paper. □

Due to Theorem 1, it suffices to generate all monotone, commutative, and associative operations $\otimes$ which are neutral with respect to 1 (greatest element of $\mathbf{L}$) and satisfy condition (ii) of Theorem 1. If $\otimes$ satisfies all these conditions, we can use (iii) to compute the residuum $\rightarrow$ of $\otimes$.

The basic idea of our algorithm is that we systematically go through all candidates $\otimes$ which may satisfy assumptions of Theorem 1 and (ii). Multiplication

$\otimes$	0	a_1	a_2	$\cdots$	a_{n-2}	1
0	0	0	0	$\cdots$	0	0
a_1	0					a_1
a_2	0					a_2
$\vdots$	$\vdots$					$\vdots$
a_{n-2}	0					a_{n-2}
1	0	a_1	a_2	$\cdots$	a_{n-2}	1

procedure *fill* (i, j)**:**
  if (1)–(3) ***not satisfied*:**
    return
  if $j \geq n-1$**:**
    set i ***to*** $i+1$
    set j ***to*** i
  if $i \geq n-1$**:**
    store $\otimes$
    return
  for each $b \in \mathrm{Bounds}(i, j)$**:**
    set $a_i \otimes a_j$ ***to*** b
    call *fill* $(i, j+1)$
  set $a_i \otimes a_j$ ***to*** "undefined"

Fig. 1. Initial assignment of truth degrees to a table for $\otimes$ and procedure *fill*

is a binary operation on L. For the sake of simplicity, assume $|L| = n$ and denote the truth degrees by $L = \{0 = a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1} = 1\}$. We assume that our indexing extends the lattice order, i.e. that $a_i \leq a_j$ implies $i \leq j$. Since for $|L| = n$, there are n^{n^2} distinct binary operations, we cannot generate all such operations before checking the desired properties. Namely, n^{n^2} is a large number even for small n. Instead, we take advantage of some of the properties to generate only reasonably small subclass of binary operations.

The task to find a multiplication can be seen as a task to fill a table as the one in Fig. 1 (left) by truth degrees. A table entry given by row i and column j represents value $a_i \otimes a_j$. Since $\otimes$ needs to be commutative, we can focus only on the upper triangle of the table (including the diagonal). Moreover, some truth degrees in the table can be fixed because from properties of residuated lattices we have

$$a \otimes 0 = 0 \otimes a = 0, \qquad a \otimes 1 = 1 \otimes a = a.$$

The other entries in the table can take any values from $L - \{1\}$. Fortunately, we need not go through all possible assignments of degrees from $L - \{1\}$ to blank entries in the table from Fig. 1 (left). We can restrict the set of possible values for each table entry using the following well-known fact:

Theorem 2. *Let* $\mathbf{L}$ *be a residuated lattice. Then, for each* $a, b \in L$*,*

(i) $a \otimes b \leq a \wedge b$;
(ii) $\bigvee\{c \otimes d \mid c, d \in L$ *such that* $c \leq a$ *and* $d \leq b\} \leq a \otimes b$. □

As we can see, Theorem 2 provides us with upper and lower bounds for the results of multiplications. In more detail, for each $a, b \in L$, the upper bound is given by (i). The lower bound can be computed using (ii) before each new assignment. Since we assign truth degrees to the table one by one, for a considered pair $a, b \in L$ of truth degrees we can take all $c, d \in L$ such that (i) the value $c \otimes d$ is already assigned, and (ii) $c \leq a$ and $d \leq b$. Then we can compute supremum of values $c \otimes d$ of all such truth degrees which is then the lower bound of $a \otimes b$.

A table for $\otimes$ is filled from its top-left corner to its bottom-right corner. Table entries are traversed in the following order: $a_1 \otimes a_1$, $a_1 \otimes a_2, \ldots, a_1 \otimes a_{n-3}$, $a_1 \otimes a_{n-2}$, $a_2 \otimes a_2$, $a_2 \otimes a_3, \ldots, a_{n-2} \otimes a_{n-2}$. For each new entry being added to the table we check several conditions to see that $\otimes$ represents a "candidate" for multiplication. Namely, for each $a, b, c \in L$ we check

$$a \otimes (b \otimes c) \;=\; (a \otimes b) \otimes c, \tag{1}$$

$$a \otimes (b \vee c) \;=\; (a \otimes b) \vee (a \otimes c), \tag{2}$$

$$a \leq b \;\text{ implies }\; a \otimes c \leq b \otimes c, \tag{3}$$

provided that the expressions (1)–(3) make sense (recall that we deal with a partial operation $\otimes$ which is being constructed, i.e., some values of $x \otimes y$ may not be defined). If the currently assigned value of $a_i \otimes a_j$ violates the conditions above, then we go back and set $a_i \otimes a_j$ to another value.

Otherwise we move to the next blank position in the table and compute possible values of the multiplication result given by Theorem 2. In more detail, for truth degrees a_i and a_j (i.e., degrees corresponding to position given by indices i and j in the table) we consider a set $\text{Bounds}(i, j) \subseteq L$ which is an interval

$$\text{Bounds}(i, j) = [b, a_i \wedge a_j]$$

where

$$b = \bigvee\{a_{\min(k,l)} \otimes a_{\max(k,l)} \,|\, (k = i \text{ and } a_l \prec a_j) \text{ or } (a_k \prec a_i \text{ and } l = j)\}$$

where $a_m \prec a_n$ denotes that a_m is covered by a_n, i.e. $a_m \leq a_n$ and $a_m \leq c \leq a_n$ implies $a_m = c$ or $a_n = c$. Then we go through all the values in $\text{Bounds}(i, j)$ and set them as the results of $a_i \otimes a_j$. Then we check (1)–(3) for $a_i \otimes a_j$ and the process continues as described above. We finish if we fill the whole table with values satisfying (1)–(3). As we have seen, the algorithm for generating $\otimes$ can be described as a recursive procedure which accepts two parameters: indices of the row and column of table Fig. 1 (left). The procedure is described in Fig. 1 (right). Note that a preliminary version of this method, which was less efficient, was described in [13].

Due to the limited scope of this paper, we postpone proof of soundness of the procedure to a full version of this paper.

3 Properties of Generated Residuated Lattices

In this section we present basic characteristics of finite residuated lattices generated by our procedure. We have used our procedure to generate all non-isomorphic residuated lattices with up to 11 elements. Prior to that, we generated all non-isomorphic lattices up to 11 elements, see [3] and studied their properties, see also [11] for a related approach.

We first focus on the numbers of n-element residuated lattices. Table 1 contains a basic summary. Columns of the table correspond to sizes of lattices (numbers of their elements). The first row contains numbers of non-isomorphic residuated lattices. The second row contains numbers of non-isomorphic linearly ordered residuated lattices (i.e., lattices with every pair of elements comparable). Note that [1] contains an error since it says that the number for $|L| = 8$ is 2368.

Table 1. Numbers of non-isomorphic finite residuated lattices up to 11 elements

	1	2	3	4	5	6	7	8	9	10	11
residuated lattices	1	1	2	7	27	142	839	5803	45466	406783	4134207
linear res. lattices	1	1	2	6	22	94	451	2386	13775	86417	590489
lattices	1	1	1	2	5	15	53	222	1078	5994	37622
res. lattice reducts	1	1	1	2	3	7	18	61	239	1125	6137

We can read from the table that small residuated lattices tend to be linear: for $|L| = 5$, 22 residuated lattices out of 27 are linear. On the other hand, with growing sizes of $|L|$, the portion of linear residuated lattices is going down: for $|L| = 11$, only one seventh of all the residuated lattices are linear.

Another interesting thing is the relationship between (numbers of) residuated lattices and (numbers of) their distinct lattice reducts. Recall that if $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is a residuated lattice, its reduct $\langle L, \wedge, \vee, 0, 1\rangle$ resulting by omitting $\otimes$ and $\rightarrow$ is a lattice. Thus, we may ask how many n-element lattices are reducts of n-element residuated lattices. This is shown in the last two rows of Table 1. The last but one row shows numbers of non-isomorphic n-element lattices [3,11]. The last row contains numbers of pairwise distinct non-isomorphic lattice reducts of all non-isomorphic residuated lattices. For instance, the values in column corresponding to $|L| = 11$ mean: there are 37622 non-isomorphic lattices but only 6137 of them can be equipped with $\otimes$ and $\rightarrow$ to form a residuated lattice. An interesting observation here is that even if the number of residuated lattices rapidly grows with growing $|L|$, the number of their lattice reducts compared to the number of all lattices (of that size) is going down. This means that with growing $|L|$, the average number of residuated lattices with the same lattice part is going up. For instance, for $|L| = 8$ the average number of residuated lattices sharing the same lattice part is approximately 95 while for $|L| = 11$ it is 673.

The values in Table 1 may suggest that most residuated lattices can be found on n-element chains. This is so for smaller residuated lattices but it is no longer true for large lattices. For instance, if we consider $|L| = 11$, we can depict the numbers of residuated lattices according to their width and height as in Table 2 (see appendices). The rows and columns in Table 2 represent heights and widths

Table 2. Numbers of 11-element residuated lattices with given heights and widths

	1	2	3	4	5	6	7	8
4	0	0	0	0	0	0	0	1
5	0	0	0	23	80	64	1883	0
6	0	0	684	38480	31280	10470	0	0
7	0	2539	113275	127288	35771	0	0	0
8	0	141182	428416	122677	0	0	0	0
9	0	825240	402523	0	0	0	0	0
10	0	1261842	0	0	0	0	0	0
11	590489	0	0	0	0	0	0	0

Table 3. Numbers of 11-element lattices with given heights and widths

	1	2	3	4	5	6	7	8	9
3	0	0	0	0	0	0	0	0	1
4	0	0	0	0	123	159	72	15	0
5	0	0	83	2212	3294	1138	126	0	0
6	0	0	2295	8464	4387	518	0	0	0
7	0	164	4413	5339	973	0	0	0	0
8	0	374	2133	805	0	0	0	0	0
9	0	217	280	0	0	0	0	0	0
10	0	36	0	0	0	0	0	0	0
11	1	0	0	0	0	0	0	0	0

Table 4. Numbers of 11-element lattice reducts with given heights and widths

	1	2	3	4	5	6	7	8
4	0	0	0	0	0	0	0	1
5	0	0	0	21	75	50	13	0
6	0	0	157	860	546	86	0	0
7	0	43	1021	1308	257	0	0	0
8	0	179	865	323	0	0	0	0
9	0	141	161	0	0	0	0	0
10	0	29	0	0	0	0	0	0
11	1	0	0	0	0	0	0	0

Table 5. Average characteristics of reducts (legend: ht/wd = avg. height/width of lattices, at = avg. number of (co)atoms, ir/pr = avg. number of irreducible/prime elements, mc/ma = avg. number of maximal chains/antichains

	1	2	3	4	5	6	7	8	9	10	11
ht	1.00	2.00	3.00	3.50	4.33	4.86	5.44	5.87	6.31	6.68	7.01
wd	1.00	1.00	1.00	1.50	1.67	2.00	2.33	2.67	3.02	3.37	3.71
at	0.00	1.00	1.00	1.50	1.33	1.71	1.89	2.10	2.23	2.38	2.50
ir	1.00	2.00	3.00	3.50	4.33	5.00	5.83	6.48	7.20	7.85	8.49
pr	1.00	2.00	3.00	3.50	4.33	4.43	4.44	4.33	4.11	3.88	3.66
mc	1.00	1.00	1.00	1.50	1.67	2.14	2.56	3.13	3.65	4.27	4.95
ma	1.00	2.00	3.00	3.50	4.33	5.00	5.67	6.48	7.38	8.47	9.76

of residuated lattices, respectively. By a *height* (or *width*) of a residuated lattice we mean the length of the longest maximal chain (or antichain) contained in its lattice part. The table entries represent numbers of distinct residuated lattices with the dimensions given by the corresponding rows and columns. Table 2 shows that most residuated lattices are defined on "high and thin lattices" but in case of $|L| = 11$, the most frequent residuated lattices are those with width 2 (see second column of Table 2). Let us mention that the distribution of all lattices and all lattice reducts according to their dimensions is quite different from that of

Table 6. Numbers of residuated lattices satisfying selected properties

	1	2	3	4	5	6	7	8	9	10	11
all res. lattices	1	1	2	7	27	142	839	5803	45466	406783	4134207
modular	1	1	2	7	27	138	775	4953	34833	269744	2303013
distributive	1	1	2	7	27	137	748	4655	31519	233186	1879285
(Π1) identity	1	1	1	4	9	51	290	2125	18165	182390	2096717
prelinear	1	1	2	7	23	100	469	2482	14256	89254	608250
(Π2) identity	1	1	1	3	8	31	156	913	6208	48054	421028
strict	1	1	1	3	7	28	142	842	5804	45473	403000
(WNM) identity	1	1	2	5	11	31	82	253	819	3064	13225
divisible	1	1	2	5	10	23	49	111	245	547	1196
involutive	1	1	1	3	3	13	17	86	185	779	2475
idempotent	1	1	1	2	3	5	8	15	26	47	80

Table 7. Numbers of selected algebras

	1	2	3	4	5	6	7	8	9	10	11
all res. lattices	1	1	2	7	27	142	839	5803	45466	406783	4134207
MTL-algebras	1	1	2	7	23	100	469	2482	14256	89254	608250
SMTL-algebras	1	1	1	3	7	24	100	472	2483	14263	89254
WNM-algebras	1	1	2	5	9	22	43	98	198	418	842
BL-algebras	1	1	2	5	9	20	38	81	161	327	645
SBL-algebras	1	1	1	3	5	10	20	41	82	166	327
IMTL-algebras	1	1	1	3	3	8	12	36	62	172	339
Heyting algebras	1	1	1	2	3	5	8	15	26	47	80
G-algebras	1	1	1	2	2	3	3	5	6	8	8
NM-algebras	1	1	1	2	1	2	1	5	4	4	3
MV-algebras	1	1	1	2	1	2	1	3	2	2	1
Π-algebras	1	1	0	1	0	0	0	1	0	0	0
ΠMTL-algebras	1	1	0	1	0	0	0	1	0	0	0

residuated lattices. Table 3 and Table 4 show the same characteristics as Table 2 for all 11-element lattices and all lattice reducts of 11-element residuated lattices, respectively. Here we can see that the numbers of lattices follow, more or less, a normal distribution (most of them have average width and height).

Table 5 shows average characteristics of the lattice reducts of residuated lattices. The rows of the table correspond to properties (see the legend in Table 5, for the notions involved we refer to [8]), the columns correspond to sizes of lattices. Table entries are the average values.

We now turn our attention to residuated lattices satisfying additional conditions. We consider the following properties of residuated lattices expressible by identities (see [2,4,5,8,9]):

(MOD) $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$ (*modularity*)
(DIS) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (*distributivity*)

(MTL) $(a \to b) \vee (b \to a) = 1$ (*prelinearity*)
(Π1) $(c \to 0) \to 0 \leq ((a \otimes c) \to (b \otimes c)) \to (a \to b)$ (*Π1-property*)
(Π2) $a \wedge (a \to 0) = 0$ (*Π2-property*)
(STR) $(a \otimes b) \to 0 = (a \to 0) \vee (b \to 0)$ (*strictness*)
(WNM) $((a \otimes b) \to 0) \vee ((a \wedge b) \to (a \otimes b)) = 1$ (*weak nilpotent minimum*)
(DIV) $a \wedge b = a \otimes (a \to b)$ (*divisibility*)
(INV) $a = (a \to 0) \to 0$ (*involution*)
(IDM) $a = a \otimes a$ (*idempotency*)

Table 6 contains numbers of residuated lattices satisfying these conditions. Table 7 summarizes numbers of algebras (particular residuated lattices) which

Table 8. Average numbers of idempotent and involutive elements in selected algebras

	1	2	3	4	5	6	7	8	9	10	11
all res. lattices	1.00	2.00	2.50	3.00	3.19	3.32	3.37	3.37	3.30	3.18	3.01
	1.00	2.00	2.50	3.14	3.37	3.79	4.07	4.39	4.68	4.98	5.26
MTL-algebras	1.00	2.00	2.50	3.00	3.17	3.42	3.58	3.74	3.87	3.97	4.01
	1.00	2.00	2.50	3.14	3.30	3.71	3.97	4.26	4.51	4.77	5.03
BL-algebras	1.00	2.00	2.50	3.20	3.67	4.25	4.74	5.31	5.80	6.33	6.82
	1.00	2.00	2.50	3.00	2.78	3.10	2.89	3.14	3.02	3.03	2.97
G-algebras	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00	11.00
	1.00	2.00	2.00	3.00	2.00	2.67	2.00	3.60	2.33	2.50	2.00
MV-algebras	1.00	2.00	2.00	3.00	2.00	3.00	2.00	4.67	3.00	3.00	2.00
	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00	11.00

Table 9. Groups of algebras sharing the same properties

	MTL-algebra	SMTL-algebra	WNM-algebra	BL-algebra	SBL-algebra	IMTL-algebra	G-algebra	NM-algebra	MV-algebra	Π-algebra	ΠMTL-algebra
3878433											
605431	×										
105952	×	×									
1551	×		×								
617	×	×		×	×						
603	×					×					
593	×			×							
36	×	×	×	×	×		×				
27	×		×	×							
18	×		×			×		×			
10	×			×		×			×		
4	×	×	×	×	×	×	×	×	×	×	×
3	×		×	×		×		×	×		

Table 10. Groups of algebras sharing the same properties (detail)

	MOD	DIS	MTL	$\Pi 1$	$\Pi 2$	STR	WEA	DIV	INV	IDE		MOD	DIS	MTL	$\Pi 1$	$\Pi 2$	STR	WEA	DIV	INV	IDE
1195367				×							617	×	×	×		×	×		×		
674576	×	×									603	×	×	×	×					×	
612291											593	×	×	×					×		
586184	×	×		×							543	×						×			
429478	×	×	×								444	×			×			×			
328566	×			×							442	×	×		×			×			
175953	×	×	×	×							355	×				×					
163065	×	×			×	×					265	×	×			×	×		×		
145596					×	×					227	×	×						×		
105952	×	×	×		×	×					221	×	×			×			×		
92801	×										146	×	×	×	×			×			
39691	×				×	×					76	×	×			×	×	×	×		×
14040					×						73	×	×			×		×	×		×
9088				×			×				38	×	×					×	×		
6413	×	×			×						36	×	×	×		×	×	×	×		×
2750							×				27	×	×	×				×	×		
2395	×	×					×				18	×	×	×	×			×		×	
1526	×			×					×		10	×	×	×	×				×	×	
1405	×	×	×				×				6	×	×		×			×		×	
742	×	×		×					×		4	×	×	×	×	×	×	×	×	×	×
652				×					×		3	×	×	×	×			×	×	×	

are defined by a combination of the above-mentioned properties. The tables show that BL-algebras are very rare among residuated lattices up to 11 elements. The situation for MTL-algebras is better but still, only 15 % of all 11-element residuated lattices are MTL-algebras. An observation which may be surprising is that ($\Pi 1$) is far more frequent a property than prelinearity (for $|L| \leq 11$).

Table 8 shows average numbers of idempotent (upper value in each table entry) and involutive (lower value in each table entry) elements of selected algebras. Here an interesting thing is that the average values for MTL-algebras are approximately the same as for all residuated lattices whereas the average values for BL-algebras are sort of opposite (MTL-algebras have more involutive elements than the idempotent ones in average while BL-algebras have more idempotent than the involutive ones).

Table 6 and Table 7 show the numbers of residuated lattices having each property but do not show, e.g., how many divisible lattices are idempotent. Such information can be found in Table 9 and in more detail in Table 10. Here, the columns denote properties considered in Table 6 and Table 7, the left-most column contains numbers of residuated lattices with given combination of properties. Each row of the tables represents one combination of properties (properties which are present are marked by “×”). Let us note that some combinations of properties are rare and some of them can be found only in larger structures. Just for illustration, the least residuated lattice which satisfies only (MOD) and ($\Pi 2$) from all the considered properties has 9 elements.

4 Conclusion and Future Research

We have presented a method for generation of finite residuated lattices up to a given size. The generated residuated lattices were used for a preliminary exploration of their quantitative properties. We have focused mainly on the exploration of numbers of various algebras (BL-algebras, MTL-algebras, ...). A database of generated structures can be found at:

`http://vychodil.inf.upol.cz/res/devel/finresl/`.

In our future work we will focus of the following topics:
- incremental algorithms that reuse previously generated structures;
- exploration of further properties of the generated residuated lattices;
- generation of structures with hedges and independent negations [5,10,17].

References

1. Bartušek T., Navara M.: Program for generating fuzzy logical operations and its use in mathematical proofs. *Kybernetika* **38**, No. 3(2002), 235–244.
2. Belohlavek R.: *Fuzzy Relational Systems: Foundations and Principles.* Kluwer Academic/Plenum Publishers, New York, 2002.
3. Belohlavek R., Vychodil V.: Scales behind computational intelligence: exploring properties of finite lattices (to appear in Proc. 1st IEEE Symposium on Foundations of Comput. Intelligence – FOCI '07, 1–5 Apr 2007, Honolulu, Hawaii, USA).
4. Esteva F., Godo L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems* **124**(2001), 271–288.
5. Esteva F., Godo L, Hájek P., Navara M.: Residuated fuzzy logics with an involutive negation. *Arch. Math. Logic* **39**(2000), 103–124.
6. Gerla G.: *Fuzzy Logic. Mathematical Tools for Approximate Reasoning.* Kluwer, Dordrecht, 2001.
7. Goguen J. A.: The logic of inexact concepts. *Synthese* **18**(1968-9), 325–373.
8. Gratzer G. A.: *General Lattice Theory.* Birkhauser (1998, 2nd ed.).
9. Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.
10. Hájek P.: On very true. *Fuzzy Sets and Systems* **124**(2001), 329–333.
11. Heitzig J., Reinhold J.: Counting Finite Lattices. *Algebra Universalis* **48**(1)(2002), 43–53.
12. Höhle U.: On the fundamentals of fuzzy set theory. *J. Math. Anal. Appl.* **201**(1996), 786–826.
13. Kure M.: *Computer-aided study of finite posets.* UP Olomouc (MSc. thesis), 2004.
14. Klir G. J., Yuan B.: *Fuzzy Sets and Fuzzy Logic. Theory and Applications.* Prentice Hall, 1995.
15. Miller G. T.: The magical number seven, plus or minus two: some limits on our capacity for processing information. *The Psychological Review* **63**(1956), 81–97.
16. Pavelka J.: On fuzzy logic I, II, III. *Z. Math. Logik Grundlagen Math.* **25**(1979), 45–52, 119–134, 447–464.
17. Vychodil V.: Truth-depressing hedges and BL-logic. *Fuzzy Sets and Systems* **157**(15)(2006), 2074–2090.