Algebras with fuzzy equalities

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1 The concept of algebra with fuzzy equality

In our previous study, we were interested in the equational fragment of first-order fuzzy logic [2, 3]. It turned out that the models of the equational fragment (which we call algebras with fuzzy equalities) are interesting and natural structures by themselves. The present paper provides an introductory study to the structures of equational fragment of fuzzy logic. The paper is an excerpt from our [4]. We aim at emphasizing two points: First, we want to introduce algebras with fuzzy equalities via natural examples. Then, after briefly going through the basic algebraic notions and (due to the limited scope of this paper almost only) commenting on how the results of the ordinary case generalize to fuzzy setting, we pay in more detail attention to the well known representation theorem by subdirectly irreducible algebras. We also comment on some future directions.

Our paper represents another attempt to application of fuzzy approach to universal algebra. Fuzzy approach to various universal algebraic concepts started with Rosenfeld's fuzzy groups [9] and has been followed by a series of papers on various fuzzy subsets of various algebras. Furthermore, our approach may be considered an alternative to so-called metric algebras [10, 11], which are basically algebras with a metric on the support set (an investigation of the relationship of metric algebras and algebras with fuzzy equalities is in progress).

For the necessary notions of fuzzy set and fuzzy logic, see [7, 8, 9]. Particularly, we use complete residuated lattices as the structures of truth values and denote them throughout the paper by $\mathbf{L}$ ($\otimes$ and $\rightarrow$ denote multiplication and residuum on $\mathbf{L}$, respectively). We put $S(A, B) = \bigwedge_{x \in X} A(x) \rightarrow B(x)$ (subsethood degree), $E(A, B) = S(A, B) \wedge S(B, A)$ (equality degree) for any sets A, B in X.

2 Definition and examples

A type is a triplet $\langle F, \approx, \sigma \rangle$ where $\approx \notin F$ and σ is a mapping $\sigma \colon F \cup \{\approx\} \rightarrow \mathbb{N} \cup \{0\}$, $\sigma(\approx) = 2$. Each $f \in F$ is called a function symbol, $\approx$ is the symbol for fuzzy equality. The mapping σ assigns the arity $\sigma(f)$ to every functional symbol $f \in F$. If there is no danger of confusion, a type will be denoted simply by F.

Definition. An *algebra with* $\mathbf{L}$*-equality* of type $\langle F, \approx, \sigma \rangle$ is a triplet $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, F^{\mathbf{A}} \rangle$ such that $\langle A, F^{\mathbf{A}} \rangle$ is an algebra of type $\langle F, \sigma \rangle$ and $\approx^{\mathbf{A}}$ is an $\mathbf{L}$-equality on A such that each $f^{\mathbf{A}} \in F^{\mathbf{A}}$ is compatible w.r.t. $\approx^{\mathbf{A}}$. The compatibility condition says that

$$(a_1 \approx^{\mathbf{A}} b_1) \otimes \cdots \otimes (a_n \approx^{\mathbf{A}} b_n) \leq \leq (f^{\mathbf{A}}(a_1, \ldots, a_n) \approx^{\mathbf{A}} f^{\mathbf{A}}(b_1, \ldots, b_n))$$

for each n-ary $f \in F$ and every $a_1, b_1, \ldots, a_n, b_n \in A$. An algebra with $\mathbf{L}$-equality will be also simply called an $\mathbf{L}$-algebra.

The concept of an $\mathbf{L}$-algebra is a natural one. First, it is obvious that the compatibility axiom expresses a natural constraint on the operations (mapping similar to similar). If one takes, e.g., $L = [0, 1]$, this constraint has a numerical character. The following are natural examples of $\mathbf{L}$-algebras.

Example. Let U be a set equipped with an $\mathbf{L}$-equivalence $\approx^U$, $A = S(U)$ be the set of all permutations of U compatible with $\approx^U$.

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The triplet $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, \circ^{\mathbf{A}} \rangle$ where $(\pi \approx^{\mathbf{A}} \sigma) = \bigwedge_u \pi(u) \approx^U \sigma(u)$ and $\circ^{\mathbf{A}}$ denotes the composition of permutations, is an **L**-algebra.

Example. Suppose we have objects (e.g. pools filled with water) and a pair of inverse operations $o^{\mathbf{A}}$ and $i^{\mathbf{A}}$ transforming these objects ($o^{\mathbf{A}}$ and $i^{\mathbf{A}}$ might be "drain one liter of water" and "pour in one liter of water"). When specifying requirements, it may be worth to take certain technical limitations into account and to require that applying $i^{\mathbf{A}}$ and $o^{\mathbf{A}}$ consecutively, one gets almost the same object. Formal specification of systems like this one can be realized using the notion of an **L**-algebra. Let A denote the set of objects (pools) and let $\approx^{\mathbf{A}}$ denote a suitable **L**-equality using which one will evaluate the condition "to get almost the same object". The **L**-algebras used to formally specify the systems are **L**-algebras $\langle A, \approx^{\mathbf{A}}, i^{\mathbf{A}}, o^{\mathbf{A}} \rangle$ (the assumption of compatibility of $i^{\mathbf{A}}$ and $o^{\mathbf{A}}$ with $\approx^{\mathbf{A}}$ is only natural). The above-mentioned requirement on the relationship between $o^{\mathbf{A}}$ and $i^{\mathbf{A}}$ translates to a formal requirement of $a \approx^{\mathbf{A}} o^{\mathbf{A}}(i^{\mathbf{A}}(a))$ being sufficiently high (e.g. at least 0.9) for each $a \in A$.

Example. Let C be an **L**-closure operator on X, let $\mathcal{S}_{\mathrm{C}} = \{A \in L^X;\, \mathrm{C}(A) = A\}$ be the system of all closed fuzzy sets of C, see [1]. That is, C is a mapping $\mathrm{C} : L^X \to L^X$ satisfying $A \subseteq \mathrm{C}(A)$, $S(A_1, A_2) \leq S(\mathrm{C}(A_1), \mathrm{C}(A_2))$, and $\mathrm{C}(A) = \mathrm{C}(\mathrm{C}(A))$, for every $A, A_1, A_2 \in L^X$. $\mathcal{S}_{\mathrm{C}}$ is a complete lattice with respect to $\subseteq$ where infima $\bigwedge$ coincide with intersections and for suprema $\bigvee$ we have $\bigvee_i A_i = \mathrm{C}(\bigcup_i A_i)$. A routine verification shows that $\langle \mathcal{S}_{\mathrm{C}}, \approx, \wedge, \vee \rangle$ is an **L**-algebra. Particularly, since the identity mapping is an **L**-closure operator C (sending A to A) for which $\mathcal{S}_{\mathrm{C}} = L^X$, we get that $\langle L^X, \approx, \cap, \cup \rangle$ is an **L**-algebra (a lattice with fuzzy equality).

3 Subalgebras, congruences, morphisms, products

Subalgebras, congruences, morphisms, and products are basic structural notions in universal algebra. We give the corresponding definitions for **L**-algebras and overview some of the results.

Definition. Let $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, F^{\mathbf{A}} \rangle$ be an **L**-algebra of type F. An **L**-algebra $\mathbf{B} = \langle B, \approx^{\mathbf{B}}, F^{\mathbf{B}} \rangle$ is called a *subalgebra* of **A**, if $B \subseteq A$, every function $f^{\mathbf{B}} \in F^{\mathbf{B}}$ is a restriction of $f^{\mathbf{A}}$ to B, and $\approx^{\mathbf{B}}$ is a restriction of $\approx^{\mathbf{A}}$ to B. A *subuniverse* of **A** is any subset $B \subseteq A$ which is closed under all operations of **A**.

As in the ordinary case, one easily gets that the collection $\mathrm{Sub}(\mathbf{A})$ of all subuniverses of **A** is closed under intersections. Therefore, we can speak of the **L**-algebra $[X]_{\mathbf{A}}$ generated by a subset $X \subseteq A$.

Definition. Let **A** be an **L**-algebra of type F. An **L**-relation θ on A is said to be a *congruence* on **A** if θ satisfies the following conditions,
(i) θ is an **L**-equivalence relation on A,
(ii) $(a \approx^{\mathbf{A}} b) \leq \theta(a, b)$,
(iii) all functions $f^{\mathbf{A}} \in F^{\mathbf{A}}$ are compatible w.r.t. θ, for arbitrary $a, b \in A$.

In the following text, the (ordinary) sets of all **L**-equivalences and congruences on an **L**-algebra **A** are denoted by $\mathrm{Eq}_{\mathbf{L}}(A)$ and $\mathrm{Con}_{\mathbf{L}}(\mathbf{A})$. Evidently, $\mathrm{Con}_{\mathbf{L}}(\mathbf{A}) \subseteq \mathrm{Eq}_{\mathbf{L}}(A)$.

Theorem 1. *$\langle \mathrm{Con}_{\mathbf{L}}(\mathbf{A}), \subseteq \rangle$ is a complete sublattice of $\langle \mathrm{Eq}_{\mathbf{L}}(A), \subseteq \rangle$.*

Proof. This proof is technically difficult but relatively straightforward, it will be included in the full version of this paper. □

A congruence is called a *principal congruence* if it is the least congruence containing a given pair $\langle x, y \rangle$ at least in a given degree $\alpha \in L$ (such a congruence is denoted by $\theta({}^{\alpha}/_{\langle x,y \rangle})$). A kernel (i.e. the 1-cut ${}^{1}\theta$) of an **L**-equivalence θ is always a bivalent equivalence relation. This enables us to define a quotient **L**-algebras.

Definition. Let θ be a congruence relation on an **L**-algebra **A**. An **L**-algebra $\mathbf{A}/\theta = \langle A/\theta, \approx^{\mathbf{A}/\theta}, F^{\mathbf{A}/\theta} \rangle$ is called a *quotient **L**-algebra of **A** modulo θ*, if
(i) $f^{\mathbf{A}/\theta}([a_1]_\theta, \ldots, [a_n]_\theta) = \left[f^{\mathbf{A}}(a_1, \ldots, a_n)\right]_\theta$ for every n-ary function $f^{\mathbf{A}/\theta} \in F^{\mathbf{A}/\theta}$ and arbitrary $a_1, \ldots, a_n \in A$,
(ii) $\left([a]_\theta \approx^{\mathbf{A}/\theta} [b]_\theta\right) = \theta(a, b)$ for all $a, b \in A$, where $[a]_\theta = \{b;\, \theta(a, b) = 1\}$ for each $a \in A$

and $A/\theta = \{[a]_\theta;\ a \in A\}$.

Definition. A mapping $h\colon \mathbf{A} \to \mathbf{B}$ is called a *homomorphism* (shortly a *morphism*), whenever $h(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(h(a_1), \dots, h(a_n))$ for every n-ary $f \in F$, $a_1, \dots, a_n \in A$ and $a \approx^{\mathbf{A}} b \leq h(a) \approx^{\mathbf{B}} h(b)$ for all $a, b \in A$.

Related notions like monomorphism, isomorphism, etc., are defined obviously. Particularly, isomorphism and embedding have to satisfy $a \approx^{\mathbf{A}} b = h(a) \approx^{\mathbf{A}} h(b)$. Moreover, $\mathrm{Ker}(h)$ is a binary $\mathbf{L}$-relation on A for which we have $\big(\mathrm{Ker}(h)\big)(a_1, a_2) = h(a_1) \approx^{\mathbf{B}} h(a_2)$.

Definition. Let I be a non-empty index set. An $\mathbf{L}$-algebra $\bigtimes_{i\in I} \mathbf{A}_i$ is called a *direct product* of a family $\{\mathbf{A}_i;\ i \in I\}$ of $\mathbf{L}$-algebras $\mathbf{A}_i = \langle A_i, \approx^{\mathbf{A}_i}, F^{\mathbf{A}_i}\rangle$ of type F whenever for every n-ary function $f^{\bigtimes_{i\in I}\mathbf{A}_i} \in F^{\bigtimes_{i\in I}\mathbf{A}_i}$ and $a_1, \dots, a_n \in \bigtimes_{i\in I} A_i$ we have $f^{\bigtimes_{i\in I}\mathbf{A}_i}(a_1, \dots, a_n)(i) = f^{\mathbf{A}_i}(a_1(i), \dots, a_n(i))$, for all $i \in I$ and the $\mathbf{L}$-equality $\approx^{\bigtimes_{i\in I}\mathbf{A}_i}$ is defined by $\big(a \approx^{\bigtimes_{i\in I}\mathbf{A}_i} b\big) = \bigwedge_{i\in I} a(i) \approx^{\mathbf{A}_i} b(i)$.

The notion of a direct product is well defined and behaves well (we have the usual properties of projections π_i, the usual properties of factor congruences, and the decomposition of finite $\mathbf{L}$-algebras into directly indecomposable ones).

4 Subdirect Irreducibility and the Representation Theorem

Definition. Let $\mathbf{A}$ be an $\mathbf{L}$-algebra of type F. The $\mathbf{L}$-algebra $\mathbf{A}$ is said to be a *subdirect product* of a family $\{\mathbf{A}_i;\ i \in I\}$ of $\mathbf{L}$-algebras of type F whenever
(i) $\mathbf{A}$ is a subalgebra of $\bigtimes_{i\in I} \mathbf{A}_i$,
(ii) $\pi_i(A) = A_i$ for every $i \in I$.
An embedding $h\colon \mathbf{A} \to \bigtimes_{i\in I} \mathbf{A}_i$ is called *subdirect* whenever $h(\mathbf{A})$ is a subdirect product of the family $\{\mathbf{A}_i;\ i \in I\}$.

Theorem 2. *If $\theta_i \in \mathrm{Con}_{\mathbf{L}}(\mathbf{A})$ for every $i \in I$ and $\bigcap_{i\in I} \theta_i = {\approx^{\mathbf{A}}}$, then the mapping $\nu\colon \mathbf{A} \to \bigtimes_{i\in I} \mathbf{A}/\theta_i$, where $\nu(a)(i) = [a]_{\theta_i}$ is a subdirect embedding. If $\mu\colon \mathbf{A} \to \bigtimes_{i\in I} \mathbf{A}_i$ is a subdirect embedding, then there is a family of congruences $\{\theta_i \in \mathrm{Con}_{\mathbf{L}}(\mathbf{A});\ i \in I\}$ such that $\bigcap_{i\in I} \theta_i = {\approx^{\mathbf{A}}}$ and $\mathbf{A}_i \cong \mathbf{A}/\theta_i$ for every $i \in I$.*

Proof. Since ν defined with $\nu(a)(i) = [a]_{\theta_i}$ is a homomorphism, one can show that ν is an embedding. Moreover, $\nu(A)(i) = \{[a]_{\theta_i};\ a \in A\} = A/\theta_i$ for every $i \in I$, thus ν is a subdirect embedding. Furthermore, let $\mu : \mathbf{A} \to \bigtimes_{i\in I} \mathbf{A}_i$ be a subdirect embedding. Put $\theta_i = \mathrm{Ker}(\mu_i)$ for every $i \in I$ (recall that $\mu_i\colon A \to A_i$, where $\mu_i(a) = \mu(a)(i)$ is a surjective homomorphism). Now, one can easily show that $\bigcap_{i\in I} \theta_i = {\approx^{\mathbf{A}}}$. Since every μ_i is surjective, a moment's reflection shows that $\mathbf{A}_i \cong \mathbf{A}/\theta_i$ for every $i \in I$. □

Definition. An $\mathbf{L}$-algebra $\mathbf{A}$ is *subdirectly irreducible* if for every subdirect embedding $h\colon \mathbf{A} \to \bigtimes_{i\in I} \mathbf{A}_i$ there is an index $j \in I$ such that $h \circ \pi_j\colon \mathbf{A} \to \mathbf{A}_j$ is an isomorphism.

Theorem 3. *An $\mathbf{L}$-algebra $\mathbf{A}$ is subdirectly irreducible iff $\mathbf{A}$ is either trivial, or there is a least congruence in $\mathrm{Con}_{\mathbf{L}}(\mathbf{A}) - \{\approx^{\mathbf{A}}\}$. In the latter case $\bigcap\big\{\mathrm{Con}_{\mathbf{L}}(\mathbf{A}) - \{\approx^{\mathbf{A}}\}\big\}$ is a principal congruence and $\langle \mathrm{Con}_{\mathbf{L}}(\mathbf{A}), \subseteq\rangle$ contains exactly one atom.*

Proof. "⇒": Suppose, by way of contradiction, that $\mathbf{A}$ is not a trivial $\mathbf{L}$-algebra and $\mathrm{Con}_{\mathbf{L}}(\mathbf{A}) - \{\approx^{\mathbf{A}}\}$ does not have the least element. Then $\bigcap\big\{\mathrm{Con}_{\mathbf{L}}(\mathbf{A}) - \{\approx^{\mathbf{A}}\}\big\} = {\approx^{\mathbf{A}}}$. Put $I = \mathrm{Con}_{\mathbf{L}}(\mathbf{A}) - \{\approx^{\mathbf{A}}\}$. Then $h\colon \mathbf{A} \to \bigtimes_{\theta_i\in I} \mathbf{A}/\theta_i$ is a subdirect embedding due to Theorem 2. Moreover, for every congruence $\theta_j \in I$ we have $\theta_j \supset {\approx^{\mathbf{A}}}$. Thus, there are $x, y \in A$ such that $x \approx^{\mathbf{A}} y < \theta_j(x, y)$, that is $x \approx^{\mathbf{A}} y < \theta_j(x, y) = \big\langle [x]_{\theta_j}, [y]_{\theta_j}\big\rangle = h_{\theta_j}(x) \approx^{\mathbf{A}/\theta_j} h_{\theta_j}(y) = (h \circ \pi_j)(x) \approx^{\mathbf{A}/\theta_j} (h \circ \pi_j)(y)$. Hence, for all $\theta_j \in I$ the mapping $h \circ \pi_j\colon \mathbf{A} \to \mathbf{A}_j$ is not an isomorphism. Consequently, the $\mathbf{L}$-algebra $\mathbf{A}$ is not subdirectly irreducible.
The converse implication is omitted, see [4].
Finally, if θ is the least congruence in $\mathrm{Con}_{\mathbf{L}}(\mathbf{A}) - \{\approx^{\mathbf{A}}\}$, then there are elements $a, b \in A$ such that $\theta(a, b) > a \approx^{\mathbf{A}} b$, so $\theta\big({}^{\theta(a,b)}/_{\langle a,b\rangle}\big) \subseteq \theta$. The converse inequality holds since $\theta\big({}^{\theta(a,b)}/_{\langle a,b\rangle}\big) > a \approx^{\mathbf{A}} b$ and θ is the least congruence with $\theta(a, b) > a \approx^{\mathbf{A}} b$. Thus, we have $\theta\big({}^{\theta(a,b)}/_{\langle a,b\rangle}\big) = \theta$, i.e. θ is a principal congruence. □

The following lemma is easy to verify.

Lemma 4. *Let $\{\theta_i;\ i \in I\}$ be a directed system of congruences, i.e. for any finite* *number* $\theta_{i_1}, \cdots, \theta_{i_k}$, $i_j \in I$ *there is* θ_i, $i \in I$ *such that* $\theta_{i_j} \subseteq \theta_i$, *for all* $j = 1, \ldots, n$. *Then* $\bigcup_{i \in I} \theta_i = \bigvee_{i \in I} \theta_i$. □

Lemma 5. *For a nontrivial* **L***-algebra* **A** *and every* $a, b \in A$ *there is a maximal* $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{A})$ *such that* $\theta(a, b) = a \approx^{\mathbf{A}} b$.

Proof. Easy by standard use of Zorn Lemma and the previous statement. □

Theorem 6. *Suppose* **A** *is a nontrivial* **L***-algebra and for every* $a, b \in A$, $a \approx^{\mathbf{A}} b < 1$ *we have*

$$\left(\bigcap_{\alpha > a \approx^{\mathbf{A}} b} \theta(\alpha/\langle a,b\rangle)\right)(a, b) > a \approx^{\mathbf{A}} b.$$

Then **A** *is isomorphic to a subdirect product of subdirectly irreducible* **L***-algebras.*

Proof. Lemma 5 yields that for every $a, b \in A$, $a \approx^{\mathbf{A}} b < 1$, the congruence $\theta_{a,b}$ is maximal w.r.t. property $\theta_{a,b}(a, b) = a \approx^{\mathbf{A}} b$. Thus, $\theta_{a,b} \vee \left(\bigcap_{\alpha > a \approx^{\mathbf{A}} b} \theta(\alpha/\langle a,b\rangle)\right)$ is the least congruence in $[\theta_{a,b}, \mathbf{A} \times \mathbf{A}] - \{\theta_{a,b}\}$. This means that $\mathbf{A}/\theta_{a,b}$ is subdirectly irreducible. Finally, using Theorem 2 and the fact $\bigcap_{a \approx^{\mathbf{A}} b < 1} \theta_{a,b} = \approx^{\mathbf{A}}$ we can conclude that there is a subdirect embedding $h\colon \mathbf{A} \to \bigtimes_{a \approx^{\mathbf{A}} b < 1} \mathbf{A}/\theta_{a,b}$, where $h(x)(\langle a, b\rangle) = [x]_{\theta_{a,b}}$. Hence, **A** is isomorphic to a subdirect product of subdirectly irreducible **L**-algebras. □

Remark. If **L** is a finite chain then the condition of Theorem 6 always holds and every **L**-algebra is then isomorphic to a subdirect product of subdirectly irreducible **L**-algebras. On the other hand, this statement is not true in general as will be shown in the following example.

Example. Take $A = \{a, b\}$ and an **L**-algebra $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, \emptyset\rangle$ (no functions). Evidently, every reflexive and symmetric **L**-relation $S\colon A \times A \to L$ such that $S(a, b) < 1$ is an **L**-equality. Indeed, every **L**-equality $\approx^{\mathbf{A}}$ on A is determined by the truth value $a \approx^{\mathbf{A}} b < 1$. Clearly, every $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{A})$ is determined by the truth value $a \approx^{\mathbf{A}} b \leq \theta(a, b)$.

From Theorem 2 and Theorem 3 it follows that **A** is isomorphic to a subdirect product iff there is a family of congruences $\{\theta_i \in \mathrm{Con}_{\mathbf{L}}(\mathbf{A});\ i \in I\}$ such that $\bigcap_{i \in I} \theta_i = \approx^{\mathbf{A}}$ and for every $i \in I$, $[\theta_i, \mathbf{A} \times \mathbf{A}]$ has exactly one atom. So suppose $\mathbf{A} \cong \bigtimes_{i \in I} \mathbf{A}/\theta_i$ and $\mathbf{L} = [0, 1]$. We will show, that there is always at least one $\mathbf{A}/\theta_i$ which is subdirectly reducible. If $\bigcap_{i \in I} \theta_i = \approx^{\mathbf{A}}$ then there is at least one index $i \in I$ such that $\theta_i(a, b) < 1$. Now let $K \subset L$ such that $\theta_i(a, b) \notin K$ and $\bigwedge K = \theta_i(a, b)$. It is possible to define congruences θ_α, $\alpha \in K$ by $\theta_\alpha(a, b) = \alpha$ (as mentioned above, θ_α is fully determined). Finally, $\bigcap_{\alpha \in K} \theta_\alpha = \theta_i(a, b)$ and $\theta_\alpha \neq \theta_i(a, b)$ for every $\alpha \in K$. Hence, $\mathbf{A}/\theta_i$ is subdirectly reducible. Altogether, $\mathbf{A} = \langle A, \approx^{\mathbf{A}}, \emptyset\rangle$ is not isomorphic to a subdirect product of subdirectly irreducible **L**-algebras.

5 Future development

Future research will be focused on further model-theoretic properties of **L**-algebras and on implicational theories and implicationally defined classes of **L**-algebras, developing further the results from [2, 3].

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