GALOIS CONNECTIONS WITH HEDGES *

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ABSTRACT: We introduce (fuzzy) Galois connections with hedges. Fuzzy Galois connections are basic structures behind so-called formal concept analysis of data with fuzzy attributes. Introducing hedges to Galois connections means introducing two parameters. The parameters influence the size of the set of all the fixpoints of a Galois connection. In the sense of formal concept analysis, the fixpoints, called formal concepts, are just the clusters extracted from data. The role of hedges is thus to control the number of extracted clusters. Stronger hedges lead to less clusters.

We present definition, examples, and basic properties of Galois connections with hedges. In addition to that, we provide their axiomatization: Galois connections with hedges are exactly mappings induced by object-attribute data with fuzzy attributes. The effect of a parameterized reduction of the number of clusters from object-attribute data is demonstrated by examples.

Keywords: Galois connection, fuzzy logic, hedge, formal concept analysis, data with fuzzy attributes

1 INTRODUCTION AND PRELIMINARIES

1.1 Introduction

Galois connections appear in several areas of mathematics and computer science, and their applications. We are interested in their applications in reasoning about object-attribute tabular data. Tabular data describing objects and their attributes represents a basic form of data. Among the several methods for analysis of object-attribute data, formal concept analysis (FCA) is becoming increasingly popular, see [11, 10]. The main aim in FCA is to extract interesting clusters (called formal concepts) from tabular data. Formal concepts correspond to maximal rectangles in a data table. The number of formal concepts in data can be large. A large collection of formal concepts is not directly comprehensible by a user. Development of methods which help to overcome the problem of large number of extracted formal concepts is thus an important task.

The aim of our paper is to introduce a new theoretical tool for formal concept analysis and reasoning about tabular data with fuzzy attributes. Our tool extends that one based on (fuzzy) Galois connections [15, 1] by introducing two additional parameters. The parameters, called (truth-stressing) hedges, are unary functions on the scale of truth degrees. Hedges can be seen as truth functions of connectives "very true". As we will see, we use hedges so that they select "important attributes" and "important objects". Stronger hedges lead to stronger criteria for "being important". The practical effect in formal concept analysis is that stronger hedges lead to smaller number of extracted clusters. This property assigns hedges a role of parameters controlling the number of extracted clusters from data.

In what follows, we present preliminary notions, definitions of Galois connections with hedges, examples and their basic properties. We show that each tabular data with fuzzy attributes induces in a natural way a Galois connection with hedges and, conversely, each Galois connections with a hedge is induced by some tabular data. Moreover, we show that our approach generalizes some of the approaches to formal concept analysis of data with fuzzy attributes proposed previously in the literature. The effect of reducing the size of extracted clusters from tabular data is demonstrated by an example.

1.2 Preliminaries

We use sets of truth degrees equipped with operations (logical connectives) so that is becomes a complete residuated lattice with a truth-stressing hedge. A complete residuated lattice with truth-stressing hedge (shortly, a hedge) [12, 13] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \le c \quad \text{iff} \quad a \le b \rightarrow c \tag{1}$$

for each $a, b, c \in L$; hedge * satisfies

$$1^* = 1, \tag{2}$$
$$a^* \le a, \tag{3}$$
$$(a \rightarrow b)^* \le a^* \rightarrow b^*, \tag{4}$$
$$a^{**} = a^*, \tag{5}$$

for each $a, b \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [12, 13]. Properties (2)–(5) have natural interpretations, e.g. (3) can be read: "if a is very true, then a is true", (4) can be read: "if $a \rightarrow b$ is very true and if a is very true, then b is very true", etc.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most

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important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:}\quad \begin{aligned} a\otimes b &= \max(a+b-1,0),\\ a\to b &= \min(1-a+b,1), \end{aligned} \tag{6}$$

$$\text{Gödel:}\quad \begin{aligned} a\otimes b &= \min(a,b),\\ a\to b &= \begin{cases} 1 & \text{if } a\le b,\\ b & \text{otherwise,}\end{cases} \end{aligned} \tag{7}$$

$$\text{Goguen (product):}\quad \begin{aligned} a\otimes b &= a\cdot b,\\ a\to b &= \begin{cases} 1 & \text{if } a\le b,\\ \frac{b}{a} & \text{otherwise.}\end{cases} \end{aligned} \tag{8}$$

In applications, we usually need a finite linearly ordered **L**. For instance, one can put $L=\{a_0=0,a_1,\dots,a_n=1\}\subseteq[0,1]$ ($a_0<\cdots<a_n$) with $\otimes$ given by $a_k\otimes a_l=a_{\max(k+l-n,0)}$ and the corresponding $\to$ given by $a_k\to a_l=a_{\min(n-k+l,n)}$. Such an **L** is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0,1]$ to L.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^*=a$ ($a\in L$); (ii) globalization [17]:

$$a^*=\begin{cases} 1 & \text{if } a=1,\\ 0 & \text{otherwise.}\end{cases} \tag{9}$$

A special case of the complete residuated lattice with hedge is the two-element Boolean algebra $\langle\{0,1\},\wedge,\vee,\otimes,\to,^*,0,1\rangle$, denoted by **2**, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge,\vee,\otimes,\to$ of **2** are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^*=0$, $1^*=1$. Note that if we prove an assertion for general **L**, then, in particular, we obtain a "crisp version" of this assertion for **L** being **2**.

Having **L**, we define usual notions: an **L**-set (fuzzy set) A in universe U is a mapping $A\colon U\to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U=\{u_1,\dots,u_n\}$ then A can be denoted by $A=\{^{a_1}/u_1,\dots,^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i=1,\dots,n$. For brevity, we introduce the following convention: we write $\{\dots,u,\dots\}$ instead of $\{\dots,^1/u,\dots\}$, and we also omit elements of U whose membership degree is zero. For example, we write $\{u,^{0.5}/v\}$ instead of $\{^1/u,^{0.5}/v,^0/w\}$, etc. Let $\mathbf{L}^U$ denote the collection of all **L**-sets in U. The operations with **L**-sets are defined componentwise. For instance, the intersection of **L**-sets $A,B\in\mathbf{L}^U$ is an **L**-set $A\cap B$ in U such that $(A\cap B)(u)=A(u)\wedge B(u)$ for each $u\in U$, etc. **2**-sets (operations with **2**-sets) can be identified with the ordinary (crisp) sets (operations with ordinary sets) of the naive set theory. Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X\times Y$.

Given $A,B\in\mathbf{L}^U$, we define a subsethood degree

$$S(A,B)=\textstyle\bigwedge_{u\in U}\big(A(u)\to B(u)\big), \tag{10}$$

which generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary **L**-relation on $\mathbf{L}^U$). Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A\subseteq B$ iff $S(A,B)=1$. As a consequence, $A\subseteq B$ iff $A(u)\le B(u)$ for each $u\in U$.

2 GALOIS CONNECTIONS WITH HEDGES

2.1 Definition of induced mappings and examples

We suppose that we are given a complete residuated lattice **L**, and two truth stressers, $*_X$ and $*_Y$ on **L**. Let X and Y be sets of objects and attributes, respectively, I be a fuzzy relation between X and Y. That is, $I: X\times Y\to L$ assigns to each $x\in X$ and each $y\in Y$ a truth degree $I(x,y)\in L$ to which object x has attribute y. The triplet $\langle X,Y,I\rangle$ represents a data table with rows and columns corresponding to objects and attributes, and table entries containing degrees $I(x,y)$.

For fuzzy sets $A\in L^X$ and $B\in L^Y$, consider fuzzy sets $A^\uparrow\in L^Y$ and $B^\downarrow\in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^\uparrow(y)=\bigwedge_{x\in X}(A(x)^{*_X}\to I(x,y)) \tag{11}$$

and

$$B^\downarrow(x)=\bigwedge_{y\in Y}(B(y)^{*_Y}\to I(x,y)). \tag{12}$$

Using basic rules of predicate fuzzy logic, $A^\uparrow(y)$ is the truth degree of "for each $x\in X$: if it is very true that x belongs to A then x has y" where "very true" is interpreted by $*_X$. Similarly for $B^\downarrow$ with "very true" interpreted by $*_Y$. That is, $A^\uparrow$ is a fuzzy set of attributes common to all objects for which it is very true that they belong to A, and $B^\downarrow$ is a fuzzy set of objects sharing all attributes for which it is very true that they belong to B. The set

$$\mathcal{B}(X^{*_X},Y^{*_Y},I)=\{\langle A,B\rangle \mid A^\uparrow=B,\ B^\downarrow=A\}$$

of all fixpoints of $\langle{}^\uparrow,{}^\downarrow\rangle$ thus contains all pairs $\langle A,B\rangle$ such that A is the collection of all objects that have all the attributes of "very B", and B is the collection of all attributes that are shared by all the objects of "very A". For the sake of brevity, we use also $\mathcal{B}(X^*,Y^*,I)$ instead of $\mathcal{B}(X^{*_X},Y^{*_Y},I)$. Also, we omit $*$ if it is the identity and write e.g. only $\mathcal{B}(X,Y^{*_Y},I)$. Given $*_X$ and $*_Y$ as parameters, elements $\langle A,B\rangle\in\mathcal{B}(X^*,Y^*,I)$ will be called formal concepts of $\langle X,Y,I\rangle$; A and B are called the extent and intent of $\langle A,B\rangle$, respectively; $\mathcal{B}(X^*,Y^*,I)$ will be called a concept lattice of $\langle X,Y,I\rangle$. Both the extent A and the intent B are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to intermediate degrees, not necessarily 0 and 1. The extent A and the intent B represent collections of objects and attributes, respectively, covered by a concept represented by $\langle A,B\rangle$.

Introduce a partial order $\le$ on $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ by putting

$$\langle A_1,B_1\rangle\le\langle A_2,B_2\rangle \text{ iff } A_1\subseteq A_2$$

for any $\langle A_1,B_1\rangle,\langle A_2,B_2\rangle\in\mathcal{B}(X^{*_X},Y^{*_Y},I)$. $\langle A_1,B_1\rangle\le\langle A_2,B_2\rangle$ means that $\langle A_2,B_2\rangle$ is more general (covers a larger collection of objects) than $\langle A_1,B_1\rangle$. The following lemma says that covering a larger collection of objects is equivalent to covering a smaller collection of attributes.

Lemma 1 *For any $\langle A_1,B_1\rangle,\langle A_2,B_2\rangle\in\mathcal{B}(X^{*_X},Y^{*_Y},I)$ we have*

$$A_1\subseteq A_2 \textit{ iff } B_2\subseteq B_1.$$

Proof. The proof follows by Theorem 3 (iii) and the fact that for $\langle A_1,B_1\rangle,\langle A_2,B_2\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$ we have $A_i^{\uparrow} = B_i$ and $B_i^{\downarrow} = A_i$. □

Remark It is easy to see that **2** (see Section 1.2) is the only residuated lattice with a hedge which has $\{0,1\}$ as its support set. This means that the only mappings $^{\uparrow}$ and $^{\downarrow}$ induced by a (crisp) relation I between X and Y as in (11) and (12) are the ordinary Galois connections. Recall that in that case, using ordinary set-theoretic notation, $A^{\uparrow} = \{y \in Y \mid \text{for each } x \in A : \langle x,y\rangle \in I\}$ and $B^{\downarrow} = \{x \in X \mid \text{for each } y \in B : \langle x,y\rangle \in I\}$, for any $A \subseteq X$ and $B \subseteq Y$.

The following are some examples that appeared previously in the literature.

Example 1 (1) Let both $*_X$ and $*_Y$ be identities on L. Then $\mathcal{B}(X,Y,I)$, i.e. $\mathcal{B}(X^*,Y^*,I)$, is what is called a (fuzzy) concept lattice, see e.g. [3, 16]. Axiomatic characterization of mappings $^{\uparrow}$ and $^{\downarrow}$ is given in [1]. A characterization of the structure of $\mathcal{B}(X,Y,I)$ is provided in [4].

(2) Recall from [9] that a crisply generated formal concept of $\langle X,Y,I\rangle$ is a formal concept $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ ($*_X$ and $*_Y$ are identities) which is generated by a crisp (fuzzy) set of attributes, i.e. there is $D \in \{0,1\}^Y$ such that $A = D^{\downarrow}$ and $B = A^{\uparrow}$. Crisply generated formal concepts may be thought of as the important ones. The number of crisply generated concepts is considerably smaller than the number of all formal concepts, see [9]. Now, it can be shown that if $*_X$ is the identity and $*_Y$ is the globalization on L, $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ is just the set of all crisply generated concepts.

(3) It can be shown (we omit details) that what is called a fuzzy concept lattice in [18] is in fact a structure isomorphic to $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ with $*_X$ and $*_Y$ being identity and globalization, respectively. If, on the other hand, $*_X$ and $*_Y$ are globalization and identity, respectively, $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ is isomorphic to what is called a one-sided fuzzy concept lattice in [14].

(4) An attribute implication [6, 8] is an expression $A \Rightarrow B$ where $A,B \in L^Y$ are fuzzy sets of attributes. The degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to which $A \Rightarrow B$ is true in $\langle X,Y,I\rangle$ is defined by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = \bigwedge_{x\in X} S(A,I_x)^* \to S(B,I_x).$$

Here, $I_x \in L^Y$ is a fuzzy set of attributes of object x, i.e. $I_x(y) = I(x,y)$, and $*$ is a truth stresser on L. Then, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is the truth degree of "each object from X having all attributes from A has also all attributes from B". It can be shown that a set T of attribute implications is a base, i.e. T semantically entails exactly the set of all attribute implications which are fully true (i.e., in degree 1) in $\langle X,Y,I\rangle$, if and only if the set of all models of T (a fuzzy set of attributes in which all implications of T are true) equals the set of all intents of formal concepts from $\mathcal{B}(X^*,Y,I)$, see [6, 8] for details.

2.2 Basic properties of induced mappings

We are going to present basic properties of operators $^{\uparrow}$ and $^{\downarrow}$. In what follows we assume that $^{\uparrow}$ and $^{\downarrow}$ are induced by (11) and (12) and that $*_X$ and $*_Y$ are hedges on a complete residuated lattice $\mathbf{L}$, i.e. satisfy (2)–(5).

For the sake of brevity we introduce mappings $\Uparrow: L^X \to L^Y$ and $\Downarrow: L^Y \to L^X$ defined by

$$A^{\Uparrow}(y) = \bigwedge_{x\in X} A(x) \to I(x,y),$$
$$B^{\Downarrow}(x) = \bigwedge_{y\in Y} B(y) \to I(x,y).$$

That is, $^{\Uparrow}$ and $^{\Downarrow}$ are induced by (11) and (12) with $*_X$ and $*_Y$ being identities on L, cf. Example 1 (1). In the following, we will use the fact that $^{\uparrow}$ is a composition of ^{*_X} and $^{\Uparrow}$ and $^{\downarrow}$ is a composition of ^{*_Y} and $^{\Downarrow}$, i.e. $A^{\uparrow} = A^{*_X\Uparrow}$ and $B^{\downarrow} = B^{*_Y\Downarrow}$ for any $A \in L^X$ and $B \in L^Y$.

We need the following lemma.

Lemma 2 *A hedge * satisfies $(\bigvee_{i\in I} a_i^*)^* = \bigvee_{i\in I} a_i^*$.*

Proof. "$\leq$" follows from (3).

"$\geq$" holds iff for any $i \in I$ we have $a_i^* \leq (\bigvee_{i\in I} a_i^*)^*$ which is true since $a_i^* = a_i^{**} \leq (\bigvee_{i\in I} a_i^*)^*$. □

Basic properties of the induced mappings follow.

Theorem 3 *Mappings $^{\uparrow}$ and $^{\downarrow}$ induced by (11) and (12) satisfy the following properties.*

(i) $S(A_1,A_2)^{*_X} \leq S(A_1^{*_X},A_2^{*_X}) \leq S(A_2^{\uparrow},A_1^{\uparrow})$,
$S(B_1,B_2)^{*_Y} \leq S(B_1^{*_Y},B_2^{*_Y}) \leq S(B_2^{\downarrow},B_1^{\downarrow})$;

(ii) $A^{*_X} \subseteq A^{\uparrow\downarrow}$ *and* $B^{*_Y} \subseteq B^{\downarrow\uparrow}$;

(iii) $A_1 \subseteq A_2$ *implies* $A_2^{\uparrow} \subseteq A_1^{\uparrow}$, *and*
$B_1 \subseteq B_2$ *implies* $B_2^{\downarrow} \subseteq B_1^{\downarrow}$;

(iv) $S(A^{*_X},B^{\downarrow}) = S(B^{*_Y},A^{\uparrow})$;

(v) *if* $(\bigvee_{j\in J} a_j)^{*_X} = \bigvee_{j\in J} a_j^{*_X}$ *then* $(\bigcup_{i\in I} A_i)^{\uparrow} = \bigcap_{i\in I} A_i^{\uparrow}$,
if $(\bigvee_{j\in J} a_j)^{*_Y} = \bigvee_{j\in J} a_j^{*_Y}$ *then* $(\bigcup_{i\in I} B_i)^{\downarrow} = \bigcap_{i\in I} B_i^{\downarrow}$;

(vi) $A^{\uparrow} = A^{*_X\uparrow}$ *and* $B^{\downarrow} = B^{*_Y\downarrow}$;

(vii) $A^{\uparrow *_Y} \subseteq A^{\uparrow\downarrow\uparrow} \subseteq A^{*_X\uparrow}$ *and* $B^{\downarrow *_X} \subseteq B^{\downarrow\uparrow\downarrow} \subseteq B^{*_Y\downarrow}$;

(viii) $(\bigcup_{i\in I} A_i^{*_X})^{\uparrow} = \bigcap_{i\in I} A_i^{\uparrow}$ *and* $(\bigcup_{i\in I} B_i^{*_Y})^{\downarrow} = \bigcap_{i\in I} B_i^{\downarrow}$;

(ix) $A^{\uparrow\downarrow} = A^{\uparrow\downarrow\uparrow\downarrow}$ *and* $B^{\downarrow\uparrow} = B^{\downarrow\uparrow\downarrow\uparrow}$.

Proof. The proof uses $A^{\uparrow} = A^{*_X\Uparrow}$ and $B^{\downarrow} = B^{*_Y\Downarrow}$, $A^{\Uparrow} \subseteq A^{\uparrow}$ and $B^{\Downarrow} \subseteq B^{\downarrow}$, and properties of $^{\Uparrow}$ and $^{\Downarrow}$ (see [1, 3]). Due to lack of space, we give only a sketch.

(i)–(iv) follow from properties of $\Uparrow$ and $\Downarrow$. For example, (ii) follows from $A^{*_X} \subseteq A^{*_X\Uparrow\Downarrow} = A^{\uparrow\Downarrow} \subseteq A^{\uparrow\downarrow}$.

(v): We have $(\bigcup_{i\in I} A_i)^{\Uparrow} = \bigcap_{i\in I} A_i^{\Uparrow}$, i.e. $\bigcap_{i\in I} A_i^{\uparrow} = \bigcap_{i\in I} A_i^{*\Uparrow} = (\bigcup_{i\in I} A_i^*)^{\Uparrow} = (\bigcup_{i\in I} A_i)^{*\Uparrow} = (\bigcup_{i\in I} A_i)^{\uparrow}$; the proof for B_i's is similar.

(vi) follows from definition of $^{\uparrow}$ and $^{\downarrow}$ using (5).

(vii) follows from (ii), (iii);

(viii) follows from (vi) properties of $^{\Uparrow}$ and $^{\Downarrow}$, and Lemma 2;

(ix) follows from (iii), (vi), and (vii): For A we have $A^{\uparrow\downarrow} = A^{\uparrow *_Y\downarrow} \supseteq A^{\uparrow\downarrow\uparrow\downarrow} \supseteq A^{*_X\uparrow\downarrow} = A^{\uparrow\downarrow}$. □

Next, we show some relationships between some properties of $^{\uparrow}$ and $^{\downarrow}$.

Theorem 4 *If mappings* $^{\uparrow}: L^X \to L^Y$ *and* $^{\downarrow}: L^Y \to L^X$ *satisfy*

$$a^{*_X} \to \{{}^{b}/y\}^{\downarrow}(x) = b^{*_Y} \to \{{}^{a}/x\}^{\uparrow}(y), \quad (13)$$

$$(\bigcup_{i\in I} A_i^{*_X})^{\uparrow} = \bigcap_{i\in I} A_i^{\uparrow}, \quad (14)$$

$$(\bigcup_{i\in I} B_i^{*_Y})^{\downarrow} = \bigcap_{i\in I} B_i^{\downarrow}, \quad (15)$$

then they also satisfy $\mathrm{S}(A^{*_X}, B^{\downarrow}) = \mathrm{S}(B^{*_Y}, A^{\uparrow})$.

Proof. We have to show $\mathrm{S}(A^{*_X}, B^{\downarrow}) \le \mathrm{S}(B^{*_Y}, A^{\uparrow})$, and $\mathrm{S}(B^{*_Y}, A^{\uparrow}) \le \mathrm{S}(A^{*_X}, B^{\downarrow})$. Due to symmetry it suffices to show only the first inequality. It is easy to see that we have to show that for any $y \in Y$ we have $\mathrm{S}(A^{*_X}, B^{\downarrow}) \otimes B^{*_Y}(y) \le A^{\uparrow}(y)$.

First, realize that (13) implies $\{{}^{1}/y\}^{\downarrow}(x) = \{{}^{1}/x\}^{\uparrow}(y)$, and (14) and (15) imply $A^{\uparrow} = A^{*_X\uparrow}$, $B^{\downarrow} = B^{*_Y\downarrow}$. So we have

$$\begin{aligned} & A^{\uparrow}(y) = A^{*_X\uparrow}(y) = (\bigcup_{x\in X}\{{}^{A(x)^{*_X}}/x\})^{\uparrow}(y) = \\ = & \bigcap_{x\in X}\{{}^{A(x)}/x\}^{\uparrow}(y) = \bigwedge_{x\in X} 1 \to \{{}^{A(x)}/x\}^{\uparrow}(y) = \\ = & \bigwedge_{x\in X} A(x)^{*_X} \to \{{}^{1}/y\}^{\downarrow}(x). \end{aligned}$$

That is, in order to show $\mathrm{S}(A^{*_X}, B^{\downarrow}) \otimes B^{*_Y}(y) \le A^{\uparrow}(y)$, it suffices to show that for any $x \in X, y \in Y$ it holds $A(x)^{*_X} \otimes \mathrm{S}(A^{*_X}, B^{\downarrow}) \otimes B^{*_Y}(y) \le \{{}^{1}/y\}^{\downarrow}(x)$, which is true. Indeed,

$$\begin{aligned} & A(x)^{*_X} \otimes \mathrm{S}(A^{*_X}, B^{\downarrow}) \otimes B^{*_Y}(y) \le \\ \le & A(x)^{*_X} \otimes (A^{*_X}(x) \to B^{\downarrow}(x)) \otimes B^{*_Y}(y) \le \\ \le & B^{\downarrow}(x) \otimes B^{*_Y}(y) \le B^{*_Y}(y) \otimes \{{}^{B(y)}/y\}^{\downarrow}(x) = \\ = & B^{*_Y}(y) \otimes (1 \to \{{}^{B(y)}/y\}^{\downarrow}(x)) = \\ = & B^{*_Y}(y) \otimes (B(y)^{*_Y} \to \{{}^{1}/x\}^{\uparrow}(y)) \le \{{}^{1}/x\}^{\uparrow}(y) = \\ = & \{{}^{1}/y\}^{\downarrow}(x). \end{aligned}$$

□

Theorem 5 *If mappings* $^{\uparrow}: L^X \to L^Y$ *and* $^{\downarrow}: L^Y \to L^X$ *satisfy* $\mathrm{S}(A^{*_X}, B^{\downarrow}) = \mathrm{S}(B^{*_Y}, A^{\uparrow})$ *then we have*

(i) $A^{*_X} \subseteq A^{\uparrow\downarrow}$ *and* $B^{*_Y} \subseteq B^{\downarrow\uparrow}$*;*

(ii) $\mathrm{S}(A_1, A_2)^{*_X} \le \mathrm{S}(A_1^{*_X}, A_2^{*_X}) \le \mathrm{S}(A_2^{\uparrow *_Y}, A_1^{\uparrow})$*, and* $\mathrm{S}(B_1, B_2)^{*_Y} \le \mathrm{S}(B_1^{*_Y}, B_2^{*_Y}) \le \mathrm{S}(B_2^{\downarrow *_X}, B_1^{\downarrow})$*;*

(iii) $A_1 \subseteq A_2$ *implies* $A_2^{\uparrow *_Y} \subseteq A_1^{\uparrow}$*, and* $B_1 \subseteq B_2$ *implies* $B_2^{\downarrow *_X} \subseteq B_1^{\downarrow}$*;*

(iv) $a^{*_X} \to \{{}^{b}/y\}^{\downarrow}(x) = b^{*_Y} \to \{{}^{a}/x\}^{\uparrow}(y)$*;*

(v) $\{{}^{1}/x\}^{\uparrow}(y) = \{{}^{1}/y\}^{\downarrow}(x)$*;*

(vi) $\{{}^{a}/x\}^{\uparrow}(y) = a^{*_X} \to \{{}^{1}/y\}^{\downarrow}(x) = a^{*_X} \to \{{}^{1}/x\}^{\uparrow}(y)$*;*

(vii) $\{{}^{b}/y\}^{\downarrow}(x) = b^{*_Y} \to \{{}^{1}/x\}^{\uparrow}(y) = b^{*_Y} \to \{{}^{1}/y\}^{\downarrow}(x)$*;*

(viii) *if* $a^{*_X} = a^{*_Y}$ *then* $\{{}^{a}/x\}^{\uparrow}(y) = \{{}^{a}/y\}^{\downarrow}(x)$*;*

(ix) $\{{}^{a^{*_X}}/x\}^{\uparrow}(y) = \{{}^{a}/x\}^{\uparrow}(y)$ *and* $\{{}^{b^{*_Y}}/y\}^{\downarrow}(x) = \{{}^{b}/y\}^{\downarrow}(x)$.

Proof. (i) holds iff $\mathrm{S}(A^{*_X}, A^{\uparrow\downarrow}) = 1$ which is true since $\mathrm{S}(A^{*_X}, A^{\uparrow\downarrow}) = \mathrm{S}(A^{\uparrow *_Y}, A^{\uparrow}) = 1$; similarly for $^{\downarrow}$;

(ii): using (i) we have $\mathrm{S}(A_1, A_2)^{*_X} \le \mathrm{S}(A_1^{*_X}, A_2^{*_X}) \le \mathrm{S}(A_1^{*_X}, A_2^{\uparrow\downarrow}) = \mathrm{S}(A_2^{\uparrow *_Y}, A_1^{\uparrow})$; similarly for $^{\downarrow}$;

(iii) follows from (ii);

(iv): we have

$$\begin{aligned} & a^{*_X} \to \{{}^{b}/y\}^{\downarrow}(x) = \mathrm{S}(\{{}^{a^{*_X}}/x\}, \{{}^{b}/y\}^{\downarrow}) = \\ = & \mathrm{S}(\{{}^{b}/y\}^{*_Y}, \{{}^{a}/x\}^{\uparrow}) = \mathrm{S}(\{{}^{b^{*_Y}}/y\}, \{{}^{a}/x\}^{\uparrow}) = \\ = & b^{*_Y} \to \{{}^{a}/x\}^{\uparrow}(y) \end{aligned}$$

(v)–(viii) follows from (iv);

(ix) follows from (vi), (vii). □

Remark Since $\mathrm{S}(A^{*_X}, B^{\downarrow}) = \mathrm{S}(B^{*_Y}, A^{\uparrow})$ is true for $^{\uparrow}$ and $^{\downarrow}$ induced by I (see Theorem 3 (iv)), all assertions of Theorem 5 are true for $^{\uparrow}$ and $^{\downarrow}$.

2.3 Axiomatization: Galois connections with hedges

Definition 6 *A Galois connection with hedges* ^{*_X} *and* ^{*_Y} *between sets* X *and* Y *is a pair* $\langle {}^{*_X}, {}^{*_Y} \rangle$ *of mappings* $^{\uparrow}: L^X \to L^Y$ *and* $^{\downarrow}: L^Y \to L^X$ *satisfying* $\mathrm{S}(A^{*_X}, B^{\downarrow}) = \mathrm{S}(B^{*_Y}, A^{\uparrow})$*,* $(\bigcup_{i\in I} A_i^{*_X})^{\uparrow} = \bigcap_{i\in I} A_i^{\uparrow}$*, and* $(\bigcup_{i\in I} B_i^{*_Y})^{\downarrow} = \bigcap_{i\in I} B_i^{\downarrow}$.

Remark It follows from Theorem 3 that mappings $^{\uparrow}$ and $^{\downarrow}$ induced by (11) and (12) form a Galois connection with hedges $*_X$ and $*_Y$.

In what follows, we denote the mappings induced by (11) and (12) by $^{\uparrow_I}$ and $^{\downarrow_I}$.

Lemma 7 *Let* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ *form a Galois connection with hedges* $*_X$ *and* $*_Y$*. Then there exists an* **L***-relation* $I \in L^{X\times Y}$ *such that* $\langle {}^{\uparrow}, {}^{\downarrow} \rangle = \langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$.

Proof. Introduce I by

$$I(x,y) = \{{}^{1}/x\}^{\uparrow}(y) = \{{}^{1}/y\}^{\downarrow}(x)$$

which is correct due to Theorem 5. Take any $A \in L^X$. We have

$$\begin{aligned} & A^{\uparrow}(y) = A^{*_X\uparrow}(y) = (\bigcup_{x\in X}\{{}^{A^{*_X}(x)}/x\})^{\uparrow}(y) = \\ = & (\bigcup_{x\in X}\{{}^{A(x)}/x\}^{*_X})^{\uparrow}(y) = (\bigcap_{x\in X}\{{}^{A(x)}/x\}^{\uparrow})(y) = \\ = & \bigwedge_{x\in X}\{{}^{A(x)}/x\}^{\uparrow}(y) = \bigwedge_{x\in X} A(x)^{*_X} \to \{{}^{1}/x\}^{\uparrow}(y) = \\ = & \bigwedge_{x\in X} A(x)^{*_X} \to I(x,y) = A^{\uparrow_I}(y), \end{aligned}$$

i.e. $^{\uparrow} = {}^{\uparrow_I}$. The proof of $^{\downarrow} = {}^{\downarrow_I}$ is similar. □

Lemma 7 says that each Galois connection with hedges is induced by some fuzzy relation. The following theorem shows that for a given hedges $*_X$ and $*_Y$, the relationship between Galois connections with $*_X$ and $*_Y$, and fuzzy relations is a bijection.

Theorem 8 *Let I be an* **L***-relation between X and Y, $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ be a Galois connection with hedges $*_X$ and $*_Y$. Then*

(i) $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$ *is a Galois connection with hedges $*_X$ and $*_Y$;*

(ii) $I_{\langle \uparrow, \downarrow \rangle}$ *defined as in the proof of Lemma 7 is an* **L***-relation between X and Y and we have*

(iii) $\langle {}^{\uparrow}, {}^{\downarrow} \rangle = \langle {}^{\uparrow_{I_{\langle \uparrow,\downarrow \rangle}}}, {}^{\downarrow_{I_{\langle \uparrow,\downarrow \rangle}}} \rangle$ *and* $I = I_{\langle \uparrow_I, \downarrow_I \rangle}$.

Proof. By Theorem 3, Theorem 5, and Lemma 7 it suffices to prove $I = I_{\langle \uparrow_I, \downarrow_I \rangle}$. We have

$$I_{\langle \uparrow_I, \downarrow_I \rangle}(x,y) = \{{}^1\!/x\}^{\uparrow_I}(y) =$$
$$= \bigwedge_{z \in X} \{{}^{1^{*_X}}\!/x\}(z) \to I(z,y) = I(x,y).$$

□

2.4 Further topics

We now briefly comment on selected topics.

2.4.1 Structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$

The structure of ordinary concept lattices is described by the so-called main theorem of concept lattices [11]. An analogy in fuzzy setting describing the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is provided in [7]. In [7], the main theorem for $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ was derived by showing that $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to some ordinary concept lattice. Among other things, this "reduction" enables us to use algorithms for concept lattices for the problem of computing $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. A direct proof of the main theorem for $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ remains an open problem (an interesting one since it may suggest new insight).

2.4.2 Closure structures

Unlike the case of fuzzy Galois connections without hedges (see [1]), the compositions of Galois connections with hedges are not fuzzy closure operators. For instance, we have only $A^{*_X} \subseteq A^{\uparrow\downarrow}$ and $B^{*_Y} \subseteq B^{\downarrow\uparrow}$, and not $A \subseteq A^{\uparrow\downarrow}$ and $B \subseteq B^{\downarrow\uparrow}$ in general (see above). It remains an open problem to study mappings satisfying the properties of the composed mappings ${}^{\uparrow\downarrow}$ and ${}^{\downarrow\uparrow}$. Note that in the ordinary case, i.e. $\mathbf{L} = \mathbf{2}$, mappings of the form ${}^{\uparrow\downarrow}$ and ${}^{\downarrow\uparrow}$ are exactly all the closure operators on X and Y, respectively.

2.4.3 Nesting of concept lattices

The following fact is remarkable and worth of further investigation. If we take $*_X$ equal the identity on L, and take two hedges $*_1$ and $*_2$ such that $*_1$ is stronger than $*_2$ (i.e. $a^{*_1} \leq a^{*_2}$ for each $a \in L$) then $\mathcal{B}(X^{*_X}, Y^{*_1}, I) \subseteq \mathcal{B}(X^{*_X}, Y^{*_2}, I)$ (we omit proof). However, for $*_X$ other than the identity, $\mathcal{B}(X^{*_X}, Y^{*_1}, I) \subseteq \mathcal{B}(X^{*_X}, Y^{*_2}, I)$ need not be the case.

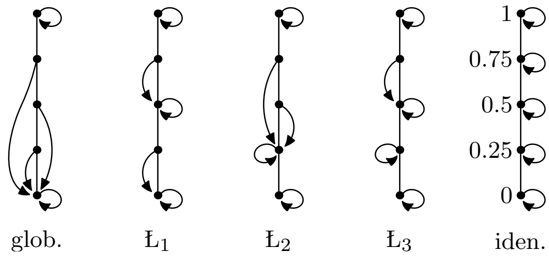

Figure 1: Truth stressers

	small	large	far	near
Mercury	1	0	0	1
Venus	0.75	0	0	1
Earth	0.75	0	0	0.75
Mars	1	0	0.5	0.75
Jupiter	0	1	0.75	0.5
Saturn	0	1	0.75	0.5
Uranus	0.25	0.5	1	0.25
Neptune	0.25	0.5	1	0
Pluto	1	0	1	0

Table 1: Data table with fuzzy attributes

2.4.4 Automatic generation of statements

In case of Galois connections with hedges, we often work with assertions which are inequalities of the form $A^{\cdots} \subseteq A^{\cdots}$ (and dually for B), where "…" stand for sequences of ${}^{\Uparrow}$, ${}^{\Downarrow}$, and *. We designed a computer program to find proofs of these assertions automatically. A database of 375 assertions (with the proofs attached) can be found at

`http://vychodil.inf.upol.cz/res/devel/aureas`.

The general inference engine is still under construction and will be available soon at the same Internet address.

3 EXAMPLES AND EXPERIMENTS

Consider a five-element Łukasiewicz chain $\mathbf{L}$ such that $L = \{0, 0.25, 0.5, 0.75, 1\}$, $\otimes$ and $\to$ given by (6). For $\mathbf{L}$, there are five truth-stressing hedges satisfying (2)–(5). That is, except for globalization and identity, there are three nontrivial hedges which will be denoted by Ł$_1$, Ł$_2$, Ł$_3$, see Fig. 1. The number of truth-stressing hedges defined on a finite chain depends on the chosen adjoint operations. For instance, for five-element Gödel chain there are eight hedges satisfying (2)–(5).

80 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	16	31	32	32	32
Ł$_1$	85	120	121	121	180
Ł$_2$	84	107	107	108	108
Ł$_3$	299	337	337	338	501
id.	560	928	637	951	1512

55 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	12	27	31	31	31
Ł$_1$	53	89	92	93	150
Ł$_2$	66	95	99	100	100
Ł$_3$	146	186	190	191	410
id.	212	448	271	540	1148

30 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	9	17	21	22	22
Ł$_1$	26	48	52	53	78
Ł$_2$	33	54	58	59	59
Ł$_3$	48	72	77	77	181
id.	59	137	91	201	425

5 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	4	7	7	8	8
Ł$_1$	8	14	15	15	17
Ł$_2$	9	15	16	16	16
Ł$_3$	10	16	17	17	31
id.	11	21	18	32	52

Table 2: Average number of clusters

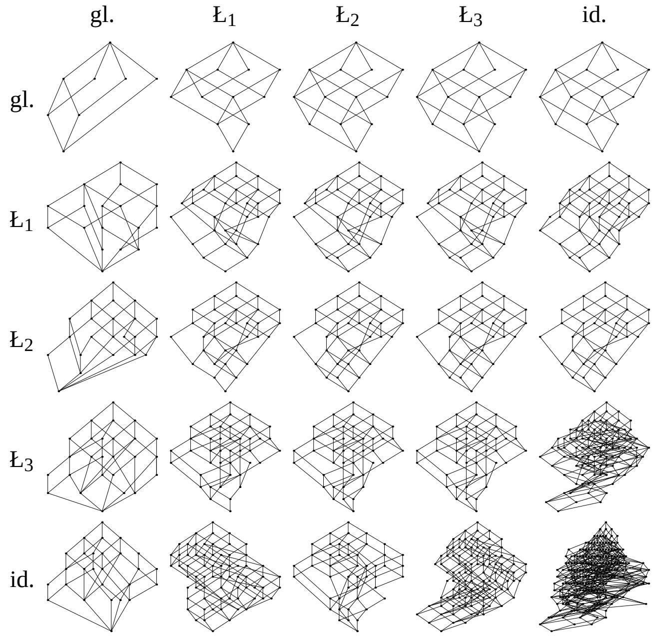

Figure 2: Concept lattices generated from data in Table 1 by all combinations of truth stressers $*_X$ and $*_Y$ from Fig. 1.

Let $\langle X, Y, I \rangle$ be given by Table 1. The set X of object consists of objects “Mercury”, “Venus”, ..., set Y contains four attributes: size of the planet (small / large), distance from the sun (far / near). Since we have five hedges on **L**, $*_X$ and $*_Y$ can be defined in 25 possible ways. Each couple $*_X$ and $*_Y$ induces a couple of operators $^{\uparrow}$ and $^{\downarrow}$. If both $*_X$ and $*_Y$ are identities then the resulting concept lattice consists of 216 clusters (formal concepts), while if $*_X$ and $*_Y$ are globalizations, the concept lattice consists of 8 clusters. These are two borderline cases. Concept lattices resulting by all the possible choices of $*_X$ and $*_Y$ are depicted in Fig. 2 (rows and columns correspond to the choice of $*_X$ and $*_Y$, respectively).

Table 2 contains a summary of average number of clusters in randomly generated data tables according to the density of input data tables: randomly generated tables have 40 objects, 5 attributes, **L** is the same structure of truth degrees as in the previous example, and the density of the generated tables varies from 5 % to 80 %. As one can see, the reduction of concept lattices using hedges as a parameter is smooth.

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