Thresholds and Shifted Attributes in Formal Concept Analysis of Data with Fuzzy Attributes*

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Abstract. We focus on two approaches to formal concept analysis (FCA) of data with fuzzy attributes recently proposed in the literature, namely, on the approach via hedges and the approach via thresholds. Both of the approaches present parameterized ways to FCA of data with fuzzy attributes. Our paper shows basic relationships between the two of the approaches. Furthermore, we show that the approaches can be combined in a natural way, i.e. we present an approach in which one deals with both thresholds and hedges. We argue that while the approach via thresholds is intuitively appealing, it can be considered a special case of the approach via hedges. An important role in this analysis is played by so-called shifts of fuzzy attributes which appeared earlier in the study of factorization of fuzzy concept lattices. In addition to fuzzy concept lattices, we consider the idea of thresholds for the treatment of attribute implications from tables with fuzzy attributes and prove basic results concerning validity and non-redundant bases.

1 Introduction and Motivation

Recently, there have been proposed several approaches to formal concept analysis (FCA) of data with fuzzy attributes, i.e. attributes which apply to objects to various degrees taken from a scale L of degrees. In particular, parameterized approaches are of interest where the parameters control the number of the extracted formal concepts. In this paper, we deal with two of these approaches, namely the approach via hedges and the approach via thresholds. Hedges were proposed as parameters for formal concept analysis of data with fuzzy attributes in [10], see also [8, 11]. For particular choices of hedges, one obtains the original approach by Pollandt and Bělohlávek [3, 23] and one-sided fuzzy approach, see [9, 22, 14]. The idea of thresholds in formal concept analysis of data with fuzzy attributes is the following. In a fuzzy setting, given a collection A of objects, the collection $A^{\uparrow}$ of all attributes shared by all objects from A is in general a fuzzy

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set, i.e. attributes y belong to $A^{\uparrow}$ in various degrees $A^{\uparrow}(y) \in L$. It is then intuitively appealing to pick a threshold δ and to consider a set ${}^{\delta}A^{\uparrow} = \{y \mid A^{\uparrow}(y) \geq \delta\}$ of all attributes which belong to $A^{\uparrow}$ in a degree greater than or equal to δ. With $\delta = 1$, this approach was proposed independently in [22, 14]. In [15], this was extended to arbitrary δ. However, the extent- and intent-forming operators defined in [15] do not form a Galois connection. This shortcoming was recognized and removed in [16] where the authors proposed new operators based on the idea of thresholds for general δ.

In our paper, we take a closer look at [16]. We show that while conceptually natural and appealing, the approach via thresholds, as proposed in [16], can be seen as a particular case of the approach via hedges. In particular, given a data with fuzzy attributes, the fuzzy concept lattices induced by the operators of [16] are isomorphic (and in fact, almost the same) to fuzzy concept lattices with hedges induced from a data containing so-called shifts of the given fuzzy attributes. This observation suggests a combination of the approaches via hedges and via thresholds which we also explore. It is interesting to note that shifts of fuzzy attributes play an important role for an efficient computation in a factorization by similarity of a fuzzy concept lattice, see [2, 7]. In addition to that, we apply the idea of thresholds to attribute implications from data with fuzzy attributes and extend some of our previous results, see e.g. [6, 12].

2 Fuzzy Concept Lattices with Hedges and Thresholds

2.1 Preliminaries from Fuzzy Logic

We first briefly recall the necessary notions from fuzzy sets and fuzzy logic (we refer to [3, 20] for further details). As a structure of truth degrees, we use an arbitrary complete residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, i.e. $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively (for instance, L is $[0, 1]$, a finite chain, etc.); $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); and $\otimes$ and $\rightarrow$ satisfy so-called adjointness property, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. Elements a of L are called truth degrees (usually, $L \subseteq [0, 1]$). $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Note that in [16], the authors do not require commutativity of $\otimes$ (but this plays no role in our note). Note that complete residuated lattices are basic structures of truth degrees used in fuzzy logic, see [18, 20]. Residuated lattices cover many structures used in applications.

For a complete residuated lattice $\mathbf{L}$, a (truth-stressing) hedge is a unary function * satisfying (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. A hedge * is a (truth function of) logical connective "very true" [21]. The largest hedge (by pointwise ordering) is identity, the least hedge is globalization which is defined by $a^* = 1$ for $a = 1$ and $a^* = 0$ for $a < 1$.

For $L = \{0, 1\}$, there exists exactly one complete residuated lattice $\mathbf{L}$ (the two-element Boolean algebra) and exactly one hedge (the identity on $\{0, 1\}$).

By $\mathbf{L}^U$ or L^U we denote the set of all fuzzy sets ($\mathbf{L}$-sets) in universe U, i.e. $L^U = \{A \mid A \text{ is a mapping of } U \text{ to } L\}$, $A(u)$ being interpreted as a degree to which u belongs to A; by 2^U we denote the set of all ordinary subsets of U, and by abuse of notation we sometimes identify ordinary subsets of U with crisp fuzzy sets from L^U, i.e. with those $A \in L^U$ for which $A(u) = 0$ or $A(u) = 1$ for each $u \in U$. For $A \in L^U$ and $a \in L$, a set ${}^aA = \{u \in U \mid A(u) \geq a\}$ is called an a-cut of A; a fuzzy set $a \to A$ in U defined by $(a \to A)(u) = a \to A(u)$ is called an a-shift of A. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \bigwedge_{u \in U} \big(A(u) \to B(u)\big),$$

which generalizes the classical subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$ (A is fully contained in B). As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

2.2 Fuzzy Concept Lattices with Hedges

A formal fuzzy context can be identified with a triplet $\langle X, Y, I\rangle$ where X is a non-empty set of objects, Y is a non-empty set of attributes, and I is a fuzzy relation between X and Y, i.e. $I : X \times Y \to L$. For $x \in X$ and $y \in Y$, a degree $I(x, y) \in L$ is interpreted as a degree to which object x has attribute y. A formal fuzzy context $\langle X, Y, I\rangle$ can be seen as a data table with fuzzy attributes with rows and columns corresponding to objects and attributes, and table entries filled with truth degrees $I(x, y)$. For $L = \{0, 1\}$, formal fuzzy contexts can be identified in an obvious way with ordinary formal contexts.

Let ^{*_X} and ^{*_Y} be hedges. For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^\uparrow \in L^Y$ and $B^\downarrow \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$ to make I explicit) defined by

$$A^\uparrow(y) = \bigwedge_{x \in X} (A^{*_X}(x) \to I(x, y)), \tag{1}$$

$$B^\downarrow(x) = \bigwedge_{y \in Y} (B^{*_Y}(y) \to I(x, y)). \tag{2}$$

Using basic rules of predicate fuzzy logic, $A^\uparrow$ is a fuzzy set of all attributes common to all objects (for which it is very true that they are) from A, and $B^\downarrow$ is a fuzzy set of all objects sharing all attributes (for which it is very true that they are) from B. The set

$$\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \mid A^\uparrow = B,\ B^\downarrow = A\}$$

of all fixpoints of $\langle {}^\uparrow, {}^\downarrow\rangle$ is called a fuzzy concept lattice of $\langle X, Y, I\rangle$; elements $\langle A, B\rangle \in \mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ will be called formal concepts of $\langle X, Y, I\rangle$; A and B are called the extent and intent of $\langle A, B\rangle$, respectively. Under a partial order $\leq$ defined on $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \ \text{ iff } \ A_1 \subseteq A_2,$$

$\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ happens to be a complete lattice and we refer to [10] for results describing the structure of $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$. Note that $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ is the basic structure used for formal concept analysis of the data table represented by $\langle X, Y, I\rangle$.

Remark 1. Operators $^{\uparrow}$ and $^{\downarrow}$ were introduced in [8, 10] as a parameterization of operators $A^{\Uparrow}(y) = \bigwedge_{x\in X}(A(x) \to I(x, y))$ and $B^{\Downarrow}(x) = \bigwedge_{y\in Y}(B(y) \to I(x, y))$ which were studied before, see [1, 4, 23]. Clearly, if both ^{*X} are ^{*Y} are identities on L, $^{\uparrow}$ and $^{\downarrow}$ coincide with $^{\Uparrow}$ and $^{\Downarrow}$, respectively. If ^{*X} or ^{*Y} is the identity on L, we omit ^{*X} or ^{*Y} in $\mathcal{B}(X^{*X}, Y^{*Y}, I)$, e.g. we write just $\mathcal{B}(X^{*X}, Y, I)$ if $^{*Y} = \mathrm{id}_L$.

2.3 Fuzzy Concept Lattices Defined by Thresholds

In addition to the pair of operators $^{\Uparrow} : L^X \to L^Y$ and $^{\Downarrow} : L^Y \to L^X$, the authors in [16] define pairs of operators (we keep the notation of [16]) $^{\star} : 2^X \to 2^Y$ and $^{\star} : 2^Y \to 2^X$, $^{\Box} : 2^X \to L^Y$ and $^{\Box} : L^Y \to 2^X$, and $^{\Diamond} : L^X \to 2^Y$ and $^{\Diamond} : 2^Y \to L^X$, as follows. Let δ be an arbitrary truth degree from L (δ plays a role of a threshold). For $A \in L^X$, $C \in 2^X$, $B \in L^Y$, $D \in 2^Y$ define $C^{\star} \in 2^Y$ and $D^{\star} \in 2^X$ by

$$C^{\star} = \{y \in Y \mid \textstyle\bigwedge_{x\in X}(C(x) \to I(x, y)) \geq \delta\}, \quad (3)$$
$$D^{\star} = \{x \in X \mid \textstyle\bigwedge_{y\in Y}(D(y) \to I(x, y)) \geq \delta\}; \quad (4)$$

$C^{\Box} \in L^Y$ and $B^{\Box} \in 2^X$ by

$$C^{\Box}(y) = \delta \to \textstyle\bigwedge_{x\in C} I(x, y), \quad (5)$$
$$B^{\Box} = \{x \in X \mid \textstyle\bigwedge_{y\in Y}(B(y) \to I(x, y)) \geq \delta\}; \quad (6)$$

and $A^{\Diamond} \in 2^Y$ and $D^{\Diamond} \in L^X$ by

$$A^{\Diamond} = \{y \in Y \mid \textstyle\bigwedge_{x\in X}(A(x) \to I(x, y)) \geq \delta\}, \quad (7)$$
$$D^{\Diamond}(x) = \delta \to \textstyle\bigwedge_{y\in D} I(x, y), \quad (8)$$

for each $x \in X$, $y \in Y$.

Denote now the corresponding set of fixpoints of these pairs of operators by

$$\mathcal{B}(X_{\star}, Y_{\star}, I) = \{\langle A, B\rangle \in 2^X \times 2^Y \mid A^{\star} = B, B^{\star} = A\},$$
$$\mathcal{B}(X_{\Box}, Y_{\Box}, I) = \{\langle A, B\rangle \in 2^X \times L^Y \mid A^{\Box} = B, B^{\Box} = A\},$$
$$\mathcal{B}(X_{\Diamond}, Y_{\Diamond}, I) = \{\langle A, B\rangle \in L^X \times 2^Y \mid A^{\Diamond} = B, B^{\Diamond} = A\},$$
$$\mathcal{B}(X_{\Uparrow}, Y_{\Downarrow}, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\Uparrow} = B, B^{\Downarrow} = A\} \quad (= \mathcal{B}(X, Y, I)).$$

2.4 Fuzzy Concept Lattices with Hedges and Thresholds

We now introduce a new pair of operators induced by a formal fuzzy context $\langle X, Y, I\rangle$. For $\delta, \varepsilon \in L$, fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow_{I,\delta}} \in L^Y$ and $B^{\downarrow_{I,\varepsilon}} \in L^X$ defined by

$$A^{\uparrow_{I,\delta}}(y) = \delta \to \textstyle\bigwedge_{x\in X}(A^{*X}(x) \to I(x, y)), \quad (9)$$
$$B^{\downarrow_{I,\varepsilon}}(x) = \varepsilon \to \textstyle\bigwedge_{y\in Y}(B^{*Y}(y) \to I(x, y)). \quad (10)$$

We will often write just $A^{\uparrow}$ and $B^{\downarrow}$ if I, δ, and ε are obvious, particularly if $\delta = \varepsilon$.

Remark 2. Note that, due to the properties of $\rightarrow$, we have that $A^{\uparrow_{I,\delta}}(y) = 1$ iff

$$\delta \leq \bigwedge_{x \in X}(A^{*_X}(x) \rightarrow I(x, y)),$$

i.e. iff the degree to which y is shared by all objects from A is at least δ. In general, $A^{\uparrow_{I,\delta}}(y)$ can be thought of as a truth degree of *the degree to which y is shared by all objects from A is at least δ.* We will show that this general approach involving the idea of thresholds subsumes the proposals of [16] as special cases. Moreover, unlike formulas (5) and (6), and (7) and (8), formulas for operators $^{\uparrow_{I,\delta}}$ and $^{\downarrow_{I,\delta}}$ are symmetric.

The set

$$\mathcal{B}(X_\delta^{*_X}, Y_\varepsilon^{*_Y}, I) = \{\langle A, B\rangle \mid A^\uparrow = B,\ B^\downarrow = A\}$$

of all fixpoints of $\langle^\uparrow, ^\downarrow\rangle$ is called a fuzzy concept lattice of $\langle X, Y, I\rangle$; elements $\langle A, B\rangle \in \mathcal{B}(X_\delta^{*_X}, Y_\varepsilon^{*_Y}, I)$ will be called formal concepts of $\langle X, Y, I\rangle$; A and B are called the extent and intent of $\langle A, B\rangle$, respectively.

Remark 3. Since $1 \rightarrow a = a$ for each $a \in L$, we have $A^{\uparrow_{I,1}} = A^{\uparrow_I}$ and $B^{\downarrow_{I,1}} = B^{\downarrow_I}$ and, therefore, $\mathcal{B}(X_1^{*_X}, Y_1^{*_Y}, I) = \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

Basic Relationships to Earlier Approaches. The following theorem shows that from a mathematical point of view, $\mathcal{B}(X_\delta^{*_X}, Y_\delta^{*_Y}, I)$ is, in fact, a fuzzy concept lattice with hedges (i.e. without thresholds) induced by a δ-shift $\delta \rightarrow I$ of I.

Theorem 1. *For any $\delta \in L$, $^{\uparrow_{I,\delta}}$ coincides with $^{\uparrow_{\delta\rightarrow I}}$, and $^{\downarrow_{I,\delta}}$ coincides with $^{\downarrow_{\delta\rightarrow I}}$. Therefore, $\mathcal{B}(X_\delta^{*_X}, Y_\delta^{*_Y}, I) = \mathcal{B}(X^{*_X}, Y^{*_Y}, \delta \rightarrow I)$.*

Proof. Using $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$ and $a \rightarrow (\bigwedge_{j\in J} b_j) = \bigwedge_{j \in J}(a \rightarrow b_j)$ we get

$$\begin{aligned} A^{\uparrow_{I,\delta}}(y) &= \delta \rightarrow \textstyle\bigwedge_{x\in X}(A^{*_X}(x) \rightarrow I(x, y)) = \\ &= \textstyle\bigwedge_{x\in X}(\delta \rightarrow (A^{*_X}(x) \rightarrow I(x, y))) = \\ &= \textstyle\bigwedge_{x\in X}(A^{*_X}(x) \rightarrow (\delta \rightarrow I(x, y))) = A^{\uparrow_{\delta\rightarrow I}}(y). \end{aligned}$$

One can proceed analogously to show that $^{\downarrow_{I,\delta}}$ coincides with $^{\downarrow_{\delta\rightarrow I}}$. Then the equality $\mathcal{B}(X_\delta^{*_X}, Y_\delta^{*_Y}, I) = \mathcal{B}(X^{*_X}, Y^{*_Y}, \delta \rightarrow I)$ follows immediately.

Remark 4. (1) Using [10], Theorem 1 yields that $\mathcal{B}(X_\delta^{*_X}, Y_\delta^{*_Y}, I)$ is a complete lattice; we show a main theorem for $\mathcal{B}(X_\delta^{*_X}, Y_\delta^{*_Y}, I)$ below.

(2) In addition to $A^{\uparrow_{I,\delta}}(y) = A^{\uparrow_{\delta\rightarrow I}}$ we also have $A^{\uparrow_{I,\delta}}(y) = (\delta \otimes A^{*_X})^{\Uparrow_I}$; similarly for $B^{\downarrow_{I,\delta}}$.

Remark 5. Note that shifted fuzzy contexts $\langle X, Y, a \rightarrow I\rangle$ play an important role in fast factorization of a fuzzy concept lattice $\mathcal{B}(X, Y, I)$ by a similarity given by a parameter a, see [2, 7]. Briefly, $\mathcal{B}(X, Y, a \rightarrow I)$ is isomorphic to a factor lattice $\mathcal{B}(X, Y, I)/^a\!\approx$ where $^a\!\approx$ is an a-cut of a fuzzy equivalence relation $\approx$ defined on $\mathcal{B}(X, Y, I)$ as in [2]. An investigation of the role of $a \rightarrow I$ in factorization of fuzzy concept lattices involving hedges is an important topic which will be a subject of a forthcoming paper.

The next theorem and Remark 6 show that the fuzzy concept lattices defined in [16] are isomorphic, and in fact identical, to fuzzy concept lattices defined by (9) and (10) with appropriate choices of ^{*X} and ^{*Y}.

Theorem 2. *Let $\mathcal{B}(X_\star, Y_\star, I)$, $\mathcal{B}(X_\Box, Y_\Box, I)$, and $\mathcal{B}(X_\Diamond, Y_\Diamond, I)$ denote the concept lattices defined in Section 2.3 using a parameter δ.*

(1) *$\mathcal{B}(X_\star, Y_\star, I)$ is isomorphic to $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$, and due to Theorem 1 also to $\mathcal{B}(X^{*X}, Y^{*Y}, \delta \to I)$, where both ^{*X} and ^{*Y} are globalizations on L.*
(2) *$\mathcal{B}(X_\Box, Y_\Box, I)$ is isomorphic to $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$, and due to Theorem 1 also to $\mathcal{B}(X^{*X}, Y^{*Y}, \delta \to I)$, where ^{*X} is globalization and ^{*Y} is the identity on L.*
(3) *$\mathcal{B}(X_\Diamond, Y_\Diamond, I)$ is isomorphic to $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$, and due to Theorem 1 also to $\mathcal{B}(X^{*X}, Y^{*Y}, \delta \to I)$, where ^{*X} is the identity and ^{*Y} is globalization on L.*

Proof. We prove only (2); the proofs for (1) and (3) are similar. First, we show that for $\langle C, D\rangle \in \mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ we have $\langle {}^1C, D\rangle \in \mathcal{B}(X_\Box, Y_\Box, I)$. Indeed, for ^{*X} being globalization we have ${}^1C = C^{*X}$ and thus

$$\begin{aligned}({}^1C)^\Box &= \delta \to \textstyle\bigwedge_{x\in {}^1C} I(x,y) = \delta \to \bigwedge_{x\in X}(({}^1C)(x) \to I(x,y)) = \\ &= \delta \to \textstyle\bigwedge_{x\in X}(C^{*X}(x) \to I(x,y)) = C^{\uparrow_{I,\delta}},\end{aligned}$$

and

$$\begin{aligned}D^\Box &= \{x \in X \mid \textstyle\bigwedge_{y\in Y}(D(y) \to I(x,y)) \geq \delta\} = \\ &= \{x \in X \mid \delta \to \textstyle\bigwedge_{y\in Y}(D(y) \to I(x,y)) = 1\} = \\ &= \{x \in X \mid D^{\downarrow_{I,\delta}}(x) = 1\} = {}^1(D^{\downarrow_{I,\delta}}) = {}^1C.\end{aligned}$$

Clearly, $\langle C, D\rangle \mapsto \langle {}^1C, D\rangle$ defines an injective mapping of $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ to $\mathcal{B}(X_\Box, Y_\Box, I)$. This mapping is also surjective. Namely, for $\langle A, B\rangle \in \mathcal{B}(X_\Box, Y_\Box, I)$ we have $\langle A^{\uparrow_{I,\delta}\downarrow_{I,\delta}}, B\rangle \in \mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ and $A = {}^1(A^{\uparrow_{I,\delta}\downarrow_{I,\delta}})$. Indeed, since $A = A^{*X}$, [8], ${}^{\uparrow_{I,\delta}} = {}^{\uparrow_{\delta\to I}}$, and ${}^{\downarrow_{I,\delta}} = {}^{\downarrow_{\delta\to I}}$ give $A^{\uparrow_{I,\delta}\downarrow_{I,\delta}\uparrow_{I,\delta}} = A^{\uparrow_{I,\delta}} = A^\Box = B$. Furthermore, $B^{\downarrow_{I,\delta}} = A^{\uparrow_{I,\delta}\downarrow_{I,\delta}}$. This shows $\langle A^{\uparrow_{I,\delta}\downarrow_{I,\delta}}, B\rangle \in \mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$. Observing

$$B^\Box = {}^\delta(B^{\downarrow_I}) = {}^1(B^{\downarrow_{\delta\to I}}) = {}^1(B^{\downarrow_{I,\delta}}) = {}^1(A^{\uparrow_{I,\delta}\downarrow_{I,\delta}})$$

finishes the proof.

Remark 6. (1) As one can see from the proof of Theorem 2, an isomorphism exists such that the corresponding elements $\langle A, B\rangle \in \mathcal{B}(X_\Box, Y_\Box, I)$ and $\langle C, D\rangle \in \mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ are almost the same, namely, $\langle A, B\rangle = \langle {}^1C, D\rangle$. A similar fact pertains to (1) and (3) of Theorem 2 as well.

(2) Alternatively, Theorem 2 can be proved using results from [11]. Consider e.g. $\mathcal{B}(X_\Box, Y_\Box, I)$: It can be shown that $\mathcal{B}(X_\Box, Y_\Box, I)$ coincides with "one-sided fuzzy concept lattice" of $\langle X, Y, \delta \to I\rangle$ (in the sense of [22]); therefore, by [11], $\mathcal{B}(X_\Box, Y_\Box, I)$ is isomorphic to a fuzzy concept lattice with hedges where ^{*X} is globalization and ^{*Y} is identity, i.e. to $\mathcal{B}(X^{*X}, Y, \delta \to I)$.

From (9) and (10) one easily obtains the following assertion.

Corollary 1. *$\mathcal{B}(X_\star, Y_\star, I)$ coincides with an ordinary concept lattice $\mathcal{B}\left(X, Y, {}^\delta I\right)$ where ${}^\delta I = \{\langle x, y\rangle \mid I(x,y) \geq \delta\}$ is the δ-cut of I.*

Remark 7. The foregoing results show that $\mathcal{B}(X_\Box, Y_\Box, I)$ and $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ are isomorphic (with appropriate ^{*X} and ^{*Y}). Moreover, $\mathcal{B}(X_\Box, Y_\Box, I)$ is almost identical to $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$, but they are not equal. Alternatively, one can proceed so as to define our operators by

$$A^{\uparrow_{I,\delta}}(y) = \left(\delta \to \textstyle\bigwedge_{x\in X}(A(x) \to I(x,y))\right)^{*Y}, \tag{11}$$

$$B^{\downarrow_{I,\varepsilon}}(x) = \left(\varepsilon \to \textstyle\bigwedge_{y\in Y}(B(y) \to I(x,y))\right)^{*X}. \tag{12}$$

Then, we even have $\mathcal{B}(X_\Box, Y_\Box, I) = \mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ (with the same choices of ^{*X} and ^{*Y}). We still prefer (9) and (10) to (11) and (12) for reasons we omit here due to lack of space.

Main Theorem of Fuzzy Concept Lattices Defined by Thresholds and Hedges. Due to Theorem 1 and Theorem 2, we can obtain main theorems for fuzzy concept lattices defined by thresholds. Omitting the proof due to lack of space, we only give here a version for the general case of $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ for the sake of illustration.

Theorem 3. (1) *$\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ is under $\leq$ a complete lattice where the infima and suprema are given by*

$$\textstyle\bigwedge_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcap_{j\in J} A_j)^{\uparrow_{I,\delta}\downarrow_{I,\delta}}, (\bigcup_{j\in J} B_j^{*Y})^{\downarrow_{I,\delta}\uparrow_{I,\delta}}\rangle, \tag{13}$$

$$\textstyle\bigvee_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcup_{j\in J} A_j^{*X})^{\uparrow_{I,\delta}\downarrow_{I,\delta}}, (\bigcap_{j\in J} B_j)^{\downarrow_{I,\delta}\uparrow_{I,\delta}}\rangle. \tag{14}$$

(2) *Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq\rangle$ is isomorphic to $\mathcal{B}(X_\delta^{*X}, Y_\delta^{*Y}, I)$ iff there are mappings $\gamma : X \times \mathrm{fix}(*_X) \to K$, $\mu : Y \times \mathrm{fix}(*_Y) \to K$ such that*

(i) *$\gamma(X \times \mathrm{fix}(*_X))$ is $\bigvee$-dense in K, $\mu(Y \times \mathrm{fix}(*_Y))$ is $\bigwedge$-dense in K;*
(ii) *$\gamma(x,a) \leq \mu(y,b)$ iff $a \otimes b \otimes \delta \leq I(x,y)$,*

with $\mathrm{fix}(*) = \{a \mid a^* = a\}$ denoting the set of all fixpoints of *.

3 Attribute Implications from Shifted Fuzzy Attributes

Let Y be a finite set of attributes (each $y \in Y$ is called an attribute). A fuzzy attribute implication (over Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ are fuzzy sets of attributes. In [6, 12, 13] we showed that (i) fuzzy attribute implications can be interpreted in data tables with fuzzy attributes (i.e., in formal fuzzy contexts); (ii) truth (validity) of fuzzy attribute implications (FAIs) in data tables with fuzzy attributes can be described as truth of implications in fuzzy concept intents; (iii) FAIs which are true in a data table with fuzzy attributes can be fully characterized by a so-called non-redundant basis of that table and the basis itself can be computed with polynomial time delay; (iv) semantic entailment from collections of fuzzy attribute implications can be characterized syntactically by an Armstrong-like set of deduction rules (two versions of completeness: characterization of FAIs which are fully entailed and characterization of degrees of entailment).

In this section we show that using the idea of thresholds one can generalize the notion of a truth of an attribute implication to a notion of δ-truth, where δ is a truth degree acting as a threshold degree. We show results answering basic questions arising with the notion of a δ-truth.

For an **L**-set $M \in \mathbf{L}^Y$ of attributes and a truth degree $\delta \in L$, define a *degree* $||A \Rightarrow B||_M^\delta \in L$ *to which* $A \Rightarrow B$ *is* δ*-true in* M by

$$||A \Rightarrow B||_M^\delta = (\delta \to S(A, M))^{*_X} \to (\delta \to S(B, M)). \tag{15}$$

Since $S(B, M)$ can be interpreted as "a degree to which M has each attribute from B", $\delta \to S(B, M)$ expresses a truth degree of proposition "a degree to which M has each attribute from B is at least δ". Thus, one can see that $||A \Rightarrow B||_M^\delta$ is interpreted as a degree to which it is true that "if it is very true that M has all attributes from A at least to degree δ, then M has all attributes from B at least to degree δ". Hence, δ acts as a threshold for antecedent and consequent of $A \Rightarrow B$ which influences the truth of $A \Rightarrow B$ in M. The notion of truth $||\cdots||_M$ being used in [6, 12, 13] is now but a particular case for $\delta = 1$, i.e. $||A \Rightarrow B||_M = ||A \Rightarrow B||_M^1$. For $\delta = 0$, which is the other borderline case, $||A \Rightarrow B||_M^0 = 1$ for each $A, B, M \in \mathbf{L}^Y$.

Theorem 4. *For each* $A, B, M \in \mathbf{L}^Y$ *and* $\delta \in L$,

$$||A \Rightarrow B||_M^\delta = ||A \Rightarrow B||_{\delta \to M}^1 = ||\delta \otimes A \Rightarrow \delta \otimes B||_M^1 = \delta \to ||\delta \otimes A \Rightarrow B||_M^1. \tag{16}$$

Proof. Using $a \to (b \to c) = b \to (a \to c)$, $a \to \bigwedge_i b_i = \bigwedge_i (a \to b_i)$, and $1 \to a = a$, see [3], one can conclude $\delta \to S(C, M) = S(C, \delta \to M) = 1 \to S(C, \delta \to M)$. Thus, $||A \Rightarrow B||_M^\delta = ||A \Rightarrow B||_{\delta \to M}^1$. The second equality follows by using $a \to (b \to c) = (a \otimes b) \to c$. The last one is also clear.

For technical reasons we introduce the following convention. For a set $\mathcal{M} \subseteq \mathbf{L}^Y$ (i.e. $\mathcal{M}$ is an ordinary set of **L**-sets) we define a degree $||A \Rightarrow B||_{\mathcal{M}}^\delta \in L$ to which $A \Rightarrow B$ is δ-true in $\mathcal{M}$ by $||A \Rightarrow B||_{\mathcal{M}}^\delta = \bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M^\delta$. Obviously,

$$||A \Rightarrow B||_{\mathcal{M}}^\delta = \bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M^\delta = \bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_{\delta \to M}^1 = ||A \Rightarrow B||_{\delta \to \mathcal{M}}^1,$$

where $\delta \to \mathcal{M} = \{\delta \to M \mid M \in \mathcal{M}\}$. For $\langle X, Y, I \rangle$, let $I_x \in \mathbf{L}^Y$ $(x \in X)$ be an **L**-set of attributes such that, for each $y \in Y$, $I_x(y) = I(x, y)$. Described verbally, I_x is the **L**-set of all attributes of object $x \in X$ in $\langle X, Y, I \rangle$. Now, a *degree* $||A \Rightarrow B||_{\langle X,Y,I \rangle}^\delta \in L$ *to which* $A \Rightarrow B$ *is* δ*-true in* (*each row of*) $\langle X, Y, I \rangle$ is defined by

$$||A \Rightarrow B||_{\langle X,Y,I \rangle}^\delta = ||A \Rightarrow B||_{\mathcal{M}}^\delta, \text{ where } \mathcal{M} = \{I_x \mid x \in X\}. \tag{17}$$

Using previous observations, we get the following

Corollary 2. *Let* $\langle X, Y, I\rangle$ *be a data table with fuzzy attributes,* $\delta \in L$*. Then*

$$||A \Rightarrow B||^{\delta}_{\langle X,Y,I\rangle} = ||A \Rightarrow B||^{1}_{\langle X,Y,\delta\to I\rangle}. \tag{18}$$

The following assertion generalizes a well-known characterization of a degree of truth of an attribute implication. It also shows that the notion of a δ-truth is well-connected to the formulas for ${}^{\uparrow_{I,\delta}}$ and ${}^{\downarrow_{I,\delta}}$.

Theorem 5. *Let* $\langle X, Y, I\rangle$ *be a data table with fuzzy attributes,* $\delta \in L$*. Then*

$$||A \Rightarrow B||^{\delta}_{\langle X,Y,I\rangle} = S(B, A^{\uparrow_{I,\delta}\downarrow_{I,\delta}}).$$

Proof. Using [12], we have $||A \Rightarrow B||^{1}_{\langle X,Y,J\rangle} = S(B, A^{\uparrow_J\downarrow_J})$ for any fuzzy relation J between X and Y. Therefore, by Theorem 1 and Corollary 2,

$$||A \Rightarrow B||^{\delta}_{\langle X,Y,I\rangle} = ||A \Rightarrow B||^{1}_{\langle X,Y,\delta\to I\rangle} = S(B, A^{\uparrow_{\delta\to I}\downarrow_{\delta\to I}}) = S(B, A^{\uparrow_{I,\delta}\downarrow_{I,\delta}}).$$

Using the concept of δ-truth, we can define appropriate notions of a model and a semantic entailment from collections of FAIs. Let T be a set of FAIs, $\delta \in L$. $M \in \mathbf{L}^Y$ is called a *δ-model* of T if $||A \Rightarrow B||^{\delta}_{M} = 1$ for each $A \Rightarrow B \in T$. The set of all δ-models of T will be denoted by $\mathrm{Mod}^{\delta}(T)$, i.e.

$$\mathrm{Mod}^{\delta}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A \Rightarrow B \in T\colon ||A \Rightarrow B||^{\delta}_{M} = 1\}. \tag{19}$$

In our terminology, models used in [6, 12, 13] are the 1-models. Using the notion of a δ-model, we define a degree of semantic δ-entailment from T. A degree $||A \Rightarrow B||^{\delta}_{T} \in L$ to which $A \Rightarrow B$ is *semantically δ-entailed from* T is defined by

$$||A \Rightarrow B||^{\delta}_{T} = ||A \Rightarrow B||^{\delta}_{\mathrm{Mod}^{\delta}(T)}. \tag{20}$$

Again, semantic 1-entailment coincides with the semantic entailment as it was introduced in [6, 12, 13]. The following assertion shows relationship between various degrees of δ-entailment.

Theorem 6. *Let* $A, B \in \mathbf{L}^Y$*,* $\delta \in L$*,* T *be a set of FAIs. Then*

(i) $\mathrm{Mod}^{\delta}(T) = \{M \in \mathbf{L}^Y \mid \delta \to M \in \mathrm{Mod}^{1}(T)\}$,
(ii) $||A \Rightarrow B||^{1}_{T} \leq ||A \Rightarrow B||^{\delta}_{T} \leq ||A \Rightarrow B||^{0}_{T}$,
(iii) $||A \Rightarrow B||^{1}_{T} = \bigwedge_{\delta\in L} ||A \Rightarrow B||^{\delta}_{T}$.

Proof. (i): By definition and using (16), $\mathrm{Mod}^{\delta}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A \Rightarrow B \in T\colon ||A \Rightarrow B||^{\delta}_{M} = 1\} = \{M \in \mathbf{L}^Y \mid \text{for each } A \Rightarrow B \in T\colon ||A \Rightarrow B||^{1}_{\delta\to M} = 1\} = \{M \in \mathbf{L}^Y \mid \delta \to M \in \mathrm{Mod}^{1}(T)\}$.

(ii): Taking into account (i), we get $||A \Rightarrow B||^{1}_{T} = \bigwedge_{M\in\mathrm{Mod}^{1}(T)} ||A \Rightarrow B||^{1}_{M} \leq \bigwedge_{\delta\to M\in\mathrm{Mod}^{1}(T)} ||A \Rightarrow B||^{1}_{\delta\to M} = \bigwedge_{M\in\mathrm{Mod}^{\delta}(T)} ||A \Rightarrow B||^{\delta}_{M} = ||A \Rightarrow B||^{\delta}_{T}$. The rest is true because $0 \to S(B, M) = 1$ for all $B, M \in \mathbf{L}^Y$.

(iii): The "$\leq$"-part follows from (ii); the "$\geq$"-part is trivial since $1 \in L$.

Remark 8. In some cases we even have

$$||A \Rightarrow B||_T^1 = ||A \Rightarrow B||_T^\delta$$

for $\delta > 0$. Inspecting the proof of Theorem 6, one can see that this is, for instance, the case when each $M \in \mathbf{L}^Y$ is of the form $M = \delta \rightarrow N$ for some $N \in \mathbf{L}^Y$. This condition is satisfied for a product structure on $[0, 1]$, i.e. when $a \otimes b = a \cdot b$. Then, $M = \delta \rightarrow (\delta \otimes M)$ as one can verify.

The following assertion shows that if $*_X$ is a globalization, then the degrees of semantic δ-entailment can be expressed as degrees of semantic 1-entailment.

Theorem 7. *Let $*_X$ be globalization. For each set T of fuzzy attribute implications and $\delta \in L$ there is a set $T' \supseteq T$ of fuzzy attribute implications such that, for each $A \Rightarrow B$,*

$$||A \Rightarrow B||_T^\delta = ||A \Rightarrow B||_{T'}^1. \tag{21}$$

Proof. Take any T and $\delta \in L$. Since $||A \Rightarrow B||_T^\delta = \bigwedge_{M \in \mathrm{Mod}^\delta(T)} ||A \Rightarrow B||_M^\delta = \bigwedge_{\delta \rightarrow M \in \mathrm{Mod}^1(T)} ||A \Rightarrow B||_{\delta \rightarrow M}^1$, it suffices to find $T' \supseteq T$ so that $\mathrm{Mod}^1(T') = \mathrm{Mod}^1(T) \cap \{\delta \rightarrow M \mid M \in \mathbf{L}^Y\}$. From [6, 12, 13] we have that $\mathrm{Mod}^1(T)$ is a closure system, i.e., an intersection of arbitrary 1-models of T is again a 1-model of T. In addition, $\bigcap_{i \in I}(\delta \rightarrow M_i) = \delta \rightarrow \bigcap_{i \in I} M_i$ is true for each $\{M_i \in \mathrm{Mod}^1(T) \mid i \in I\}$ from which we get that $\mathcal{M}_\delta = \mathrm{Mod}^1(T) \cap \{\delta \rightarrow M \mid M \in \mathbf{L}^Y\}$ is closed under arbitrary intersections. Thus, for each $M \in \mathbf{L}^Y$ let $cl_\delta(M) \in \mathbf{L}^Y$ denote the least fuzzy set of attributes (w.r.t. "$\subseteq$") which belongs to $\mathcal{M}_\delta$. Moreover, put $T' = T \cup \{M \Rightarrow cl_\delta(M) \mid M \in \mathbf{L}^Y\}$. Clearly, $\mathrm{Mod}^1(T') \subseteq \mathcal{M}_\delta$ because $T \subseteq T'$, and for each $M \in \mathrm{Mod}^1(T')$ there is $N \in \mathbf{L}^Y$ such that $M = \delta \rightarrow N$ (the existence of N follows from the fact that M is a 1-model of $\{M \Rightarrow cl_\delta(M) \mid M \in \mathbf{L}^Y\}$, i.e., it belongs to $\{\delta \rightarrow M \mid M \in \mathbf{L}^Y\}$). The "$\supseteq$"-part is true because if $M \notin \mathrm{Mod}^1(T')$, then either $M \notin \mathrm{Mod}^1(T)$ or there is $N \in \mathbf{L}^Y$ such that $||N \Rightarrow cl_\delta(N)||_M^1 \neq 1$ from which we further obtain $N \subseteq M$ and $cl_\delta(N) \not\subseteq M$ yielding $M \notin \{\delta \rightarrow M \mid M \in \mathbf{L}^Y\}$. In either case, assuming $M \notin \mathrm{Mod}^1(T')$, we get $M \notin \mathcal{M}_\delta$. Finally, $||A \Rightarrow B||_T^\delta = \bigwedge_{\delta \rightarrow M \in \mathrm{Mod}^1(T)} ||A \Rightarrow B||_{\delta \rightarrow M}^1 = \bigwedge_{M \in \mathrm{Mod}^1(T')} ||A \Rightarrow B||_M^1 = ||A \Rightarrow B||_{T'}^1$.

We now turn our attention to particular sets of FAIs which describe δ-truth of attribute implications in a given data table via semantic entailment. Let $\langle X, Y, I\rangle$ be a data table with fuzzy attributes, $\delta \in L$ be a truth degree. A set T of FAIs is called *δ-complete in $\langle X, Y, I\rangle$* if, for each $A \Rightarrow B$, $||A \Rightarrow B||_T^1 = ||A \Rightarrow B||_{\langle X,Y,I\rangle}^\delta$. If T is δ-complete and no proper subset of T is δ-complete, then T is called a *non-redundant δ-basis of $\langle X, Y, I\rangle$*. The following assertion gives a criterion of δ-completeness.

Theorem 8. *Let $\langle X, Y, I\rangle$ be a data table with fuzzy attributes, $\delta \in L$, $*_Y$ be identity. Then T is δ-complete in $\langle X, Y, I\rangle$ iff* $\mathrm{Mod}^1(T) = \mathrm{Int}(X_\delta^{*_X}, Y_\delta^{*_Y}, I)$.

Proof. By definition, we get that T is δ-complete in $\langle X, Y, I\rangle$ iff, for each $A \Rightarrow B$, $||A \Rightarrow B||_T^1 = ||A \Rightarrow B||^{\delta}_{\langle X,Y,I\rangle}$, which is true iff $||A \Rightarrow B||_T^1 = ||A \Rightarrow B||^1_{\langle X,Y,\delta\rightarrow I\rangle}$, i.e., iff T is 1-complete in $\langle X, Y, \delta \rightarrow I\rangle$. The latter is true, by results on 1-completeness [6, 12], if and only if $\mathrm{Mod}^1(T) = \mathrm{Int}(X^{*_X}, Y^{*_Y}, \delta \rightarrow I)$. By Theorem 1, $\mathrm{Int}(X^{*_X}, Y^{*_Y}, \delta \rightarrow I) = \mathrm{Int}(X_\delta^{*_X}, Y_\delta^{*_Y}, I)$, finishing the proof.

	size		distance	
	small (s)	large (l)	far (f)	near (n)
Mercury	1	0	0	1
Venus	0.75	0	0	1
Earth	0.75	0	0	0.75
Mars	1	0	0.5	0.75
Jupiter	0	1	0.75	0.5
Saturn	0	1	0.75	0.5
Uranus	0.25	0.5	1	0.25
Neptune	0.25	0.5	1	0
Pluto	1	0	1	0

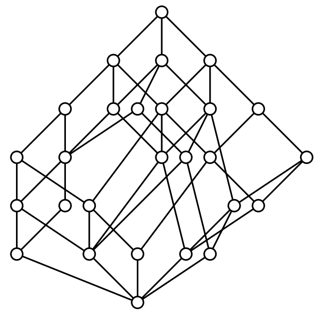


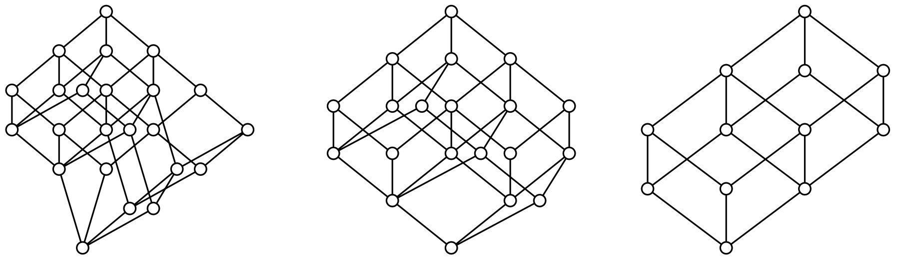
Fig. 1. Data table with fuzzy attributes and fuzzy concept lattice

Remark 9. (1) Theorem 8 says that a set T of FAIs which is δ-complete in a given data table with fuzzy attributes not only describes truth of all FAIs in the table, but also fully determines the corresponding concept lattice (intents of $\mathcal{B}(X_\delta^{*_X}, Y_\delta^{*_Y}, I)$ are exactly the models of T). More importantly, the claim was proven due to existing results on FAIs and due to a reduction of the problem of δ-completeness to the problem of 1-completeness.

(2) Previous results [6, 12] allow us to determine a non-redundant basis of a data table with fuzzy attributes. The procedure is the following. Given $\langle X, Y, I\rangle$ and $\delta \in L$, first determine $\langle X, Y, \delta \rightarrow I\rangle$, then find a non-redundant basis T of $\langle X, Y, \delta \rightarrow I\rangle$ (in the sense of [6, 12]) which is, in consequence, a non-redundant δ-basis of $\langle X, Y, I\rangle$. Note that the well-known Guigues-Duquenne basis [17, 19] is a particular case of the above-described basis for $\mathbf{L} = \mathbf{2}$ and $\delta = 1$.

4 Illustrative Example

Take a finite Łukasiewicz chain $\mathbf{L}$ with $L = \{0, 0.25, 0.5, 0.75, 1\}$ as a structure of truth degrees. Consider an input data table $\langle X, Y, I\rangle$ depicted in Fig 1 (left) which describes properties of planets of our solar system. The set X of object consists of objects "Mercury", "Venus", ..., set Y contains four attributes: size of the planet (small / large), distance from the sun (far / near). Let $*_X$ be globalization and $*_Y$ be identity. Fuzzy concept lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ (i.e., $\mathcal{B}(X_1^{*_X}, Y_1^{*_Y}, I)$) is depicted in Fig.1 (right). A non-redundant (minimal) basis (i.e., 1-basis) of $\langle X, Y, I\rangle$ consists of the following fuzzy attribute implications.



$\{{}^{0.75}/s, {}^{0.75}/l, f\} \Rightarrow \{l, n\}$
$\{{}^{0.75}/s, {}^{0.5}/n\} \Rightarrow \{n\}$
$\{{}^{0.75}/s\} \Rightarrow \{s\}$
$\{{}^{0.5}/s, {}^{0.75}/n\} \Rightarrow \{s, n\}$
$\{l\} \Rightarrow \{{}^{0.75}/n\}$
$\{{}^{0.5}/l, {}^{0.75}/n\} \Rightarrow \{l\}$
$\{{}^{0.5}/l\} \Rightarrow \{{}^{0.75}/l, f\}$
$\{f, {}^{0.5}/n\} \Rightarrow \{{}^{0.75}/l\}$
$\{{}^{0.5}/f\} \Rightarrow \{{}^{0.75}/f\}$
$\{n\} \Rightarrow \{s\}$
$\{\} \Rightarrow \{{}^{0.25}/s, {}^{0.25}/l, {}^{0.25}/f, {}^{0.25}/n\}$

$\{s, {}^{0.75}/l\} \Rightarrow \{n\}$
$\{s, {}^{0.75}/n\} \Rightarrow \{n\}$
$\{{}^{0.75}/s, n\} \Rightarrow \{s\}$
$\{{}^{0.75}/l\} \Rightarrow \{l, f\}$
$\{{}^{0.75}/f\} \Rightarrow \{f\}$
$\{\} \Rightarrow \{{}^{0.5}/s, {}^{0.5}/l, {}^{0.5}/f, {}^{0.5}/n\}$

$\{l\} \Rightarrow \{f\}$
$\{\} \Rightarrow \{{}^{0.75}/s, {}^{0.75}/l, {}^{0.75}/f, {}^{0.75}/n\}$

Fig. 2. Fuzzy concept lattices and corresponding non-redundant bases

$\{s, {}^{0.5}/l, f\} \Rightarrow \{l, n\}$
$\{s, {}^{0.5}/f, n\} \Rightarrow \{l, f\}$
$\{{}^{0.75}/s, {}^{0.5}/f\} \Rightarrow \{s\}$
$\{{}^{0.75}/s, {}^{0.25}/n\} \Rightarrow \{{}^{0.75}/n\}$
$\{{}^{0.5}/s\} \Rightarrow \{{}^{0.75}/s\}$
$\{{}^{0.25}/s, {}^{0.75}/f\} \Rightarrow \{f\}$
$\{{}^{0.25}/s, {}^{0.5}/n\} \Rightarrow \{{}^{0.75}/s, {}^{0.75}/n\}$

$\{{}^{0.75}/l\} \Rightarrow \{l, {}^{0.5}/n\}$
$\{{}^{0.25}/l, {}^{0.5}/n\} \Rightarrow \{l\}$
$\{{}^{0.25}/l\} \Rightarrow \{{}^{0.5}/l, {}^{0.75}/f\}$
$\{f\} \Rightarrow \{{}^{0.25}/s\}$
$\{{}^{0.75}/f, {}^{0.25}/n\} \Rightarrow \{{}^{0.5}/l\}$
$\{{}^{0.25}/f\} \Rightarrow \{{}^{0.5}/f\}$
$\{{}^{0.75}/n\} \Rightarrow \{{}^{0.75}/s\}$

Models of the basis are exactly the intents of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, see [6, 12]. We now show how the fuzzy concept lattice and its minimal basis change when we consider thresholds $\delta \in L$. Recall that if $\delta = 1$, then $\mathcal{B}(X_1^{*_X}, Y_1^{*_Y}, I) = \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, and a 1-basis of $\langle X, Y, I\rangle$ is the previous set of FAIs. For $\delta = 0$ the concept lattice is trivial (one-element) and the basis consists of a single fuzzy attribute implication $\{\} \Rightarrow \{s, l, f, n\}$. Fig. 2 (left) depicts fuzzy concept lattice $\mathcal{B}(X_{0.75}^{*_X}, Y_{0.75}^{*_Y}, I)$ and its non-redundant basis (below the lattice); Fig. 2 (middle) depicts fuzzy concept lattice $\mathcal{B}(X_{0.5}^{*_X}, Y_{0.5}^{*_Y}, I)$ and the corresponding basis. Finally, Fig. 2 (right) depicts $\mathcal{B}(X_{0.25}^{*_X}, Y_{0.25}^{*_Y}, I)$ and its basis.

5 Conclusions

We showed that the extent- and intent-forming operators from [16], based on the idea of thresholds, form, in fact, a particular case of Galois connections with hedges. Furthermore, we showed that the formulas for Galois connections with hedges can be extended using the idea of thresholds and that this extension still reduces to the original formulas. This enables us to reduce the problems of Galois connections with hedges and thresholds and their concept lattices to problems of Galois connections with hedges and their concept lattices. Nevertheless, the concept of Galois connections with hedges and thresholds is intuitively appealing, the thresholds being parameters which influence the size of the resulting concept lattices. In addition to that, we introduced thresholds to the definition of truth of fuzzy attribute implication and proved some results concerning reduction to the case without thresholds and some further results.

Further research will deal with the following problems:

- the role of shifted attributes in FCA of data with fuzzy attributes,
- analysis of the relationship between δ_1 and δ_2, and the corresponding structures $\mathcal{B}\left(X_{\delta_1}^{*_X}, Y_{\delta_1}^{*_Y}, I\right)$ and $\mathcal{B}\left(X_{\delta_2}^{*_X}, Y_{\delta_2}^{*_Y}, I\right)$,
- further investigation of thresholds in fuzzy attribute implications.

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