Generating non-redundant bases of data tables with fuzzy attributes by reduction of complete sets of fuzzy attribute implications

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Abstract— Presented is indirect method for generating non-redundant bases of data tables with fuzzy attributes. Fuzzy attribute implications (FAIs) are formulas describing particular dependencies of attributes in data. Non-redundant bases are minimal sets of FAIs describing all FAIs which are true (valid) in given data. Our method is based on reducing sets of FAIs describing all dependencies which can contain redundant FAIs. By removing the redundant FAIs we obtain this way non-redundant bases. We show that the procedure can generate smaller bases than previous methods. We present new theoretical results, the algorithm, its complexity analysis, and statistics demonstrating efficiency of the method.

I. INTRODUCTION

This paper is a continuation of a series of papers [3]–[13] in which we have focused on generating non-redundant bases of data tables with fuzzy attributes and related problems. Data tables with fuzzy attributes represent a basic form of tabular data describing graded relationship between objects and attributes. In a more detail, a data table with fuzzy attributes can be seen as a table with rows corresponding to objects, columns corresponding to attributes, and table entries indicating degrees to which objects have attributes: table entry $I(x, y)$ corresponding to object x and attribute y expresses a degree to which "object x has attribute y". The notion of a data table with fuzzy attributes is indeed a basic one since each fuzzy relation between two finite domains can be seen as such a table. The concept of a data table with fuzzy attributes can easily be understood by users as a means of expressing their observations about objects and their attributes (properties).

A data table with fuzzy attributes can contain a hidden information (e.g., a dependency between degrees to which objects have attributes) which is not apparent at first sight even to a user who has constructed the table. In this paper we are interested in discovering such hidden information. In particular, we focus on if-then dependencies between attributes which may be valid in the table. The dependencies will be described using if-then rules called fuzzy attribute implications (FAIs, see [3], [4] and [20] for a related approach). A fuzzy attribute implication can be seen as a rule $A \Rightarrow B$ saying, roughly speaking, if an object has all the attributes from A, then it has all the attributes from B. Intuitively, in each data table there are many dependencies which are true (i.e., are satisfied by all objects in the table), however, a lot of them are trivial or follow from other dependencies.

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Therefore, we wish to describe all FAIs which are true in a given data table by a small set of the most important ones.

Particular sets of important FAIs describing all dependencies in data, called non-redundant bases of data tables with fuzzy attributes, have been studied earlier. In [3], [10], we presented theoretical foundations for non-redundant bases and direct methods for their computation. We have shown that our ability to quickly generate a non-redundant basis depends on our choice of a so-called truth-stressing hedge (technical details explained later). If the hedge is a so-called globalization [21], there is an efficient way to compute non-redundant bases which are in addition minimal (with respect to the number of FAIs). For general hedges, however, up till now there is no satisfactory direct algorithm. In this paper, we develop indirect procedure which has partially been proposed in [11]. This procedure enables us to generate non-redundant bases for general hedges. Bases (for general hedges) produced by our new procedure can be smaller than bases considered in [3], [10].

II. PRELIMINARIES

A. Fuzzy Logic and Fuzzy Sets

We introduce complete residuated lattices with hedges, which are used as structures of truth degrees, and related notions. A complete residuated lattice with a (truth-stressing) hedge [2], [17], [18] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$; hedge * satisfies, for each $a, b \in L$, (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [17], [18]. Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization [21]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For hedge *, we denote by $\mathrm{fix}(*)$ the set of all its fixed points, i.e. $\mathrm{fix}(*)$ is defined by

$$\mathrm{fix}(*) = \{a \in L \mid a^* = a\}. \tag{2}$$

From properties of * we get that $\mathrm{fix}(*) = \{a^* \mid a \in L\}$. Sometimes, we will use multiple hedges on $\mathbf{L}$ and we will denote them by ${}^*, {}^\bullet, {}^\diamond, \ldots$ Since globalization will play an

important role, let us assume that symbol "•" always denotes a globalization, i.e., a hedge defined by (1).

A usual choice of **L** is a structure with $L = [0, 1]$ (unit interval with its genuine ordering), $\otimes$ being a left-continuous (or continuous) t-norm with the corresponding $\rightarrow$. Recall that important subclasses of residuated lattices can be defined by identities. For instance, BL-algebras [17] are residuated lattices satisfying, for each $a, b \in L$, $(a \rightarrow b) \vee (b \rightarrow a) = 1$ (prelinearity) and $a \wedge b = a \otimes (a \rightarrow b)$ (divisibility). In this paper we will be interested in computational aspects of generating if-then rules from data. This topic includes important problems connected with computational tractability. As a consequence of having tractable procedures, in some cases we will be compelled to restrict ourselves only to finite residuated lattices (we will comment on this later on). Let us note that a special case of a (finite) complete residuated lattice with hedge is the two-element Boolean algebra which is the structure of truth degrees of the classical logic, see [2], [17] for details.

With **L** acting as a structure of truth degrees, we define usual notions: an **L**-set (fuzzy set) A in universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all **L**-sets in U. If $U = \{u_1, \ldots, u_n\}$ then we denote $A \in \mathbf{L}^U$ as usual by an expression $A = \{^{A(u_1)}/u_1, \ldots, ^{A(u_n)}/u_n\}$ in which we possibly skip pairs $^0/u_i$ (zero membership degree in A) and write just u_j instead of $^1/u_j$ (full membership in A). Operations with **L**-sets are defined componentwise, see e.g. [2], [19]. For $a \in L$ and $A \in \mathbf{L}^U$ we define **L**-set $a \otimes A \in \mathbf{L}^U$ (so-called a-multiple of A) by $(a \otimes A)(u) = a \otimes A(u)$. For $A \in \mathbf{L}^U$ and hedge * we define $A^* \in \mathbf{L}^U$ by $(A(u))^*$. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \bigwedge_{u \in U} \big(A(u) \rightarrow B(u) \big), \tag{3}$$

which generalizes the classical subsethood relation $\subseteq$. Truth degree $S(A, B) \in L$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$ (i.e., if A is fully contained in B). Using properties of residuated lattices, one can show that $A \subseteq B$ iff, for each $u \in U$, $A(u) \leq B(u)$, see [2]. Binary **L**-relations (binary fuzzy relations) between U and V can be thought of as **L**-sets in the universe $U \times V$. That is, a binary **L**-relation $R \in \mathbf{L}^{U \times V}$ between a set U and a set V is a mapping assigning to each $u \in U$ and each $v \in V$ a truth degree $R(u, v) \in L$ (a degree to which u and v are related by R).

In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [2], [17]. Throughout the rest of the paper, **L** denotes an arbitrary complete residuated lattice with hedge *.

B. Fuzzy Attribute Implications

This section present a survey of basic notions of fuzzy attribute implications, more details can be found in [3], [4], [5], [8]. Let Y be a finite set of elements called attributes. A *fuzzy attribute implication* (*over Y*) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). The intended meaning of $A \Rightarrow B$ is: "if it is (very) true

Country	A	P	G
Austria	0.5	0	0.5
Belgium	0.5	0	0.5
Czech Rep.	0	0	0.25
Denmark	0	0	0.5
Finland	0	0	0.5
France	1	0.75	0.5
Germany	0.75	1	0.5
Greece	0	0	0.25
Hungary	0	0	0.25
Ireland	0	0	0.5
Italy	0.5	0.75	0.5

Country	A	P	G
Luxembourg	0	0	1
Netherlands	0.25	0.25	0.5
Norway	0	0	0.5
Poland	0.25	0.5	0.25
Portugal	0	0	0.25
Romania	0	0.25	0
Slovakia	0	0	0.25
Slovenia	0	0	0.25
Spain	0	0.5	0.5
Sweden	0.5	0	0.5
UK	0.5	0.75	0.5

Fig. 1. Data table (legend: $A =$ high number of asylum applications; $P =$ large population; $G =$ high Gross Domestic Product per capita)

that an object has all attributes from A, then it has also all attributes from B". Fuzzy attribute implications (FAIs) are basic formulas of fuzzy attribute logic (FAL). FAIs are meant to be interpreted in data tables with fuzzy attributes [2], [3], [4]. A data table with fuzzy attributes can be seen as a triplet $\langle X, Y, I \rangle$ where X is a finite set of objects, Y is a finite set of attributes (the same as above in the definition of a fuzzy attribute implication), and $I \in \mathbf{L}^{X \times Y}$ is a binary **L**-relation between X and Y assigning to each object $x \in X$ and each attribute $y \in Y$ a degree $I(x, y)$ to which x has y. The triplet $\langle X, Y, I \rangle$ can be seen as a table with rows and columns corresponding to objects $x \in X$ and attributes $y \in Y$, respectively, and table entries containing degrees $I(x, y)$.

Example 1: An example of a data table with fuzzy attributes is in Fig. 1. In this case, objects from X are selected European countries, attributes from Y are selected socio-economic indicators of countries (see legend), and table entries contain degrees to which the indicators apply to countries. The data was taken from the CIA World Factbook 2006[1] and EuroStat[2] and scaled to graded attributes using a discrete scale of truth degrees $0 < 0.25 < 0.5 < 0.75 < 1$.

A row of a table $\langle X, Y, I \rangle$ corresponding to an object $x \in X$ can be seen as a fuzzy set $I_x \in \mathbf{L}^Y$ of attributes to which an attribute $y \in Y$ belongs to a degree $I_x(y) = I(x, y)$. For fuzzy set $M \in \mathbf{L}^Y$ of attributes (think of M as of a fuzzy set representing a row in some data table with fuzzy attributes), we define a *degree* $||A \Rightarrow B||^*_M \in L$ *to which $A \Rightarrow B$ is true in M* by

$$||A \Rightarrow B||^*_M = S(A, M)^* \rightarrow S(B, M). \tag{4}$$

If * (hedge) is clear form context, we write just $||A \Rightarrow B||_M$ instead of $||A \Rightarrow B||^*_M$. Clearly, if $M = I_x$, i.e. if M is the fuzzy set of attributes of object x in data table $\langle X, Y, I \rangle$, then $||A \Rightarrow B||_M$ is the degree to which "if it is (very) true that x has all attributes from A then x has all attributes from B". Thus, (4) can be seen as a natural definition of truth (semantic validity) of a FAI in a single row of a data table with fuzzy attributes. Two things that need to be justified in a more detail are (i) the role of * in definition (4), and (ii) why A, B in $A \Rightarrow B$ are considered as fuzzy sets instead

[1]http://www.cia.gov/cia/publications/factbook/
[2]http://epp.eurostat.ec.europa.eu

of ordinary sets. Ad (i): our general definition which uses $*$ captures two important ways to define truth of FAIs which are now thanks to hedges approached in a single theory. Namely, if $*$ is globalization, then $||A \Rightarrow B||_M^* = 1$ is equivalent to "if $A \subseteq M$, then $B \subseteq M$" (i.e., only full containments in M are taken into account). On the other hand if $*$ is identity, then $||A \Rightarrow B||_M^* = 1$ is equivalent to "$S(A, M) \leq S(B, M)$" (subsethood degrees in M are taken into account). Thus, hedge $*$ can be seen as a "parameter" influencing the semantics of FAIs. Ad (ii): rules with fuzzy sets of attributes have greater expressive power than rules which use just ordinary sets of attributes. Moreover, degrees to which attributes belong to $A, B \in \mathbf{L}^Y$ can naturally be interpreted as thresholds which seems to be advantageous, see [7], [8].

We now extend this definition to "all rows" of the data table as follows. For $\langle X, Y, I \rangle$ and $A \Rightarrow B$ we put

$$||A \Rightarrow B||_{\langle X,Y,I \rangle} = \bigwedge_{x \in X} ||A \Rightarrow B||_{I_x}. \qquad (5)$$

Degree $||A \Rightarrow B||_{\langle X,Y,I \rangle}$ will be called a *degree to which $A \Rightarrow B$ is true in $\langle X, Y, I \rangle$*. From (5) we can read that $||A \Rightarrow B||_{\langle X,Y,I \rangle}$ is defined as a degree to which it is true that "for each object $x \in X$: if it is (very) true that x has all attributes from A, then x has all attributes from B", i.e. it indeed represents a degree to which $A \Rightarrow B$ is true in "all rows" of the data table. As mentioned in the introduction, we will be looking for a concise description of all FAIs which are true in given data tables.

C. Semantic and Syntactic Entailment

First we will be interested in entailment of FAIs. In fuzzy attribute logic we distinguish two basic types of entailment: a semantic one (based on "truth in models", see [3], [4]) and a syntactic one (based on particular notions of provability, see [6], [7], [9]). In this paper we will take advantage of the semantic one.

Let T be a set of fuzzy attribute implications. Fuzzy set $M \in \mathbf{L}^Y$ is called a *model of T (under $*$)* if, for each FAI $A \Rightarrow B \in T$, we have $||A \Rightarrow B||_M^* = 1$. The set of all models of T is denoted by $\text{Mod}(T)$ or by $\text{Mod}^*(T)$ if we wish to denote hedge $*$ explicitly. A degree $||A \Rightarrow B||_T^*$ to which $A \Rightarrow B$ *semantically follows from T (under $*$)* is defined by

$$||A \Rightarrow B||_T^* = \bigwedge_{M \in \text{Mod}^*(T)} ||A \Rightarrow B||_M^*. \qquad (6)$$

Again, we omit all occurrences of $*$ in (6) if $*$ is obvious. Observe that entailment degree $||A \Rightarrow B||_T$ is defined as a degree to which "$A \Rightarrow B$ is true in each model of T". If $||A \Rightarrow B||_T = 1$, we say that $A \Rightarrow B$ *is fully entailed by T*. Entailment degrees $||\cdots||_T^*$ can be characterized using the least models [11]. In a more detail, each system $\text{Mod}^*(T)$ of models is closed under arbitrary intersections, i.e. for each $M \in \mathbf{L}^Y$ we can consider the least model of T (the least with respect to "$\subseteq$") containing M. Such a model will be denoted by $cl_T^*(M)$ and will be called a *model of T generated by M*. From [11] we have the following observation:

Theorem 2 (see [11]): For each set T and $A \Rightarrow B$,

$$||A \Rightarrow B||_T^* = S(B, cl_T^*(A)). \qquad \blacksquare$$

Let us mention that semantic entailment from FAIs can be axiomatized in both ordinary (axiomatization of fully entailed formulas) and graded style (axiomatization of graded consequence) using a deductive system of Armstrong-like [1] deduction rules, see [6], [7], [9], and a survey paper [8].

D. Completeness and Non-redundancy

In this section we introduce particular sets of FAIs describing, via semantic entailment, all FAIs which are true in given data tables. We will need the following notions.

T is called *complete in $\langle X, Y, I \rangle$* if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I \rangle}$, i.e. if, for each $A \Rightarrow B$, a degree to which T entails $A \Rightarrow B$ coincides with a degree to which $A \Rightarrow B$ is true in $\langle X, Y, I \rangle$. By definition, complete sets fully characterize truth of FAIs in data tables. Since we wish to have a concise representation of all true FAIs, we focus on complete sets which are "as least as possible", hence the following notion: If T is complete in $\langle X, Y, I \rangle$ and no FAI from T is fully entailed by the other FAIs in T, then T is called a *non-redundant basis of $\langle X, Y, I \rangle$*. Due to observations from [6], T is a non-redundant basis of $\langle X, Y, I \rangle$ iff T is complete in $\langle X, Y, I \rangle$ and no proper subset of T is complete in $\langle X, Y, I \rangle$.

In this paper we develop an indirect method of generating non-redundant bases which starts with a "reasonably small" complete set of FAIs, which may be redundant, and reduces the set to a non-redundant basis by removing FAIs which are entailed by the other ones. In order to introduce the initial complete sets, we need some more notions.

Each data table with fuzzy attributes $\langle X, Y, I \rangle$ induces a couple of operators $\Uparrow \colon \mathbf{L}^X \to \mathbf{L}^Y$, $\Downarrow \colon \mathbf{L}^Y \to \mathbf{L}^X$ defined by

$$A^\Uparrow(y) = \bigwedge_{x \in X} \big(A(x) \to I(x, y) \big), \qquad (7)$$
$$B^\Downarrow(x) = \bigwedge_{y \in Y} \big(B(y) \to I(x, y) \big). \qquad (8)$$

Operators $\Downarrow, \Uparrow$ form so-called (fuzzy) Galois connection [2], [14], [15] and fixed points of the operators identify important conceptual clusters hidden in $\langle X, Y, I \rangle$ (see [2] for details). Furthermore, we consider a system $\mathcal{P} \subseteq \mathbf{L}^Y$ of fuzzy sets of attributes containing all $P \in \mathbf{L}^Y$ satisfying the following conditions:

(i) $P \neq P^{\Downarrow * \Uparrow}$, and
(ii) for each $Q \in \mathcal{P}$ such that $Q \subset P$: $Q^{\Downarrow * \Uparrow} \subseteq P$.

System $\mathcal{P}$ defined by (i) and (ii) is inspired by [16]. If $\mathbf{L}$ is finite, which will be assumed from this moment on, system $\mathcal{P}$ is uniquely given (for infinite $\mathbf{L}$, $\mathcal{P}$ may not exist [4]; this is why we need finite structures of truth degrees). In addition to that, $\mathcal{P}$ determines an important complete set of FAIs:

Theorem 3 (see [4], [11]): *Let $\langle X, Y, I \rangle$ be a data table with fuzzy attributes. Then $T = \{P \Rightarrow P^{\Downarrow * \Uparrow} \mid P \in \mathcal{P}\}$ is complete in $\langle X, Y, I \rangle$. If in addition $*$ is globalization then T is a non-redundant basis of $\langle X, Y, I \rangle$.* $\blacksquare$

III. Getting Non-redundant Bases by Reduction

Due to Theorem 3, for each $\langle X, Y, I\rangle$, we can get a set T of FAIs which is complete in $\langle X, Y, I\rangle$. From the computational point of view, T can be determined by an algorithm which works with polynomial time delay (see [3], [11]). For general hedges, it can happen that T contains FAIs which are entailed by other ones. When this situation occurs, T is not a non-redundant basis of $\langle X, Y, I\rangle$. In this section we focus on this problem. We take T produced by Theorem 3 and propose a procedure that reduces T to a non-redundant basis. We first present theoretical insight into the problem and then we introduce a new algorithm.

Call $A \Rightarrow B$ *redundant in* T (*under* *) if $A \Rightarrow B \in T$ and $\|A \Rightarrow B\|_{T-\{A\Rightarrow B\}}^* = 1$. Due to Theorem 2, we get that $A \Rightarrow B \in T$ is redundant in T if $B \subseteq cl_{T-\{A\Rightarrow B\}}^*(A)$ (i.e., iff B is contained in the least model of $T - \{A \Rightarrow B\}$ under * generated by A). This observation alone is not suitable to test redundancy of $A \Rightarrow B$ because the computational cost of determining $cl_{T-\{A\Rightarrow B\}}^*(A)$ directly is too high. On the other hand, in case of globalization, the least models generated by fuzzy sets of attributes can be computed with asymptotic complexity $O(n)$ using algorithm called GLinClosure, see [12]. It is then appealing to reduce the problem of finding $cl_{T-\{A\Rightarrow B\}}^*(A)$ to the problem of finding $cl_{T'}^\bullet(A)$, where "$\bullet$" is globalization and T' is a set of FAIs which is (somehow) determined from T. Then we will be able to use the efficient algorithm from [12]. In what follows we show that this approach is indeed possible.

Let T be a set of FAIs, * be any hedge, $\bullet$ be globalization. Define a set T^* of FAIs as follows:

$$T^* = \{c\otimes A \Rightarrow c\otimes B \mid A \Rightarrow B \in T, \, c \in \text{fix}(*),$$
$$\text{and } c\otimes B \nsubseteq c\otimes A\}. \qquad (9)$$

The condition $c\otimes B \nsubseteq c\otimes A$, which appears in (9), is equivalent to saying that fuzzy attribute implication $c\otimes A \Rightarrow c\otimes B$ is not an instance of one of the axioms of fuzzy attribute logic. Instances of any axiom of FAL need not be added to T^* because they automatically follow from any set of FAIs, see [6], [7], [9] for a detailed explanation. The following assertions show important properties of sets defined by (9).

Lemma 4: *For each fuzzy attribute implication* $A \Rightarrow B$ *over* Y *and* $M \in \mathbf{L}^Y$*, we have*

$$\|A \Rightarrow B\|_M^* = 1 \text{ iff } M \in \text{Mod}^\bullet(\{A \Rightarrow B\}^*).$$

Proof: Notice that $\{A \Rightarrow B\}^*$ represents T^* as defined in (9) for T being $\{A \Rightarrow B\}$. Since $M \in \text{Mod}^\bullet(\{A \Rightarrow B\}^*)$ is true iff, for each $c \in \text{fix}(*)$, $\|c\otimes A \Rightarrow c\otimes B\|_M^\bullet = 1$, it suffices to check that $\|A \Rightarrow B\|_M^* = 1$ iff, for each $c \in \text{fix}(*)$, we have:

$$\text{if } c\otimes A \subseteq M \text{ then } c\otimes B \subseteq M. \qquad (10)$$

Let $\|A \Rightarrow B\|_M^* = 1$ and assume that $c\otimes A \subseteq M$ for $c \in \text{fix}(*)$. We have $c\otimes A \subseteq M$ iff, for each $y \in Y$, $c \otimes A(y) \leq M(y)$ which is iff, for each $y \in Y$, $c \leq A(y) \rightarrow M(y)$, which is iff $c \leq S(A, M)$. Furthermore, by definition of

$\|\cdots\|_M^*$ from $\|A \Rightarrow B\|_M^* = 1$ we get that $S(A, M)^* \leq S(B, M)$. Hence, monotony and idempotency of * yield

$$c = c^* \leq S(A, M)^* \leq S(B, M),$$

which is equivalent to $c\otimes B \subseteq M$, showing (10).

Conversely, let (10) be true for each $c \in \text{fix}(*)$. Since $S(A, M)^* \in \text{fix}(*)$, from (10) we get that if $S(A, M)^* \otimes A \subseteq M$ then $S(A, M)^* \otimes B \subseteq M$. But $S(A, M)^* \otimes A \subseteq M$ is true. Indeed, using adjointness property, we get that $S(A, M)^* \otimes A \subseteq M$ is equivalent to $S(A, M)^* \leq S(A, M)$ which is true due to subdiagonality of *. Thus, from (10) being true for $c = S(A, M)^*$ we derive $S(A, M)^* \otimes B \subseteq M$, which is equivalent to $S(A, M)^* \leq S(B, M)$, i.e. it is equivalent to $\|A \Rightarrow B\|_M^* = 1$. $\blacksquare$

Theorem 5: *Let* $\mathbf{L}$ *be a complete residuated lattice with hedge* **. Then for each set* T *of fuzzy attribute implications,*

(i) $\text{Mod}^*(T) = \text{Mod}^\bullet(T^*)$,

(ii) $\|A \Rightarrow B\|_T^* = S(B, cl_{T^*}^\bullet(A))$,

(iii) $\|A \Rightarrow B\|_T^* = 1$ *iff* $B \subseteq cl_{T^*}^\bullet(A)$.

Proof: (i) is a consequence of Lemma 4. In a more detail, $M \in \text{Mod}^*(T)$ iff, for each $A \Rightarrow B \in T$, $\|A \Rightarrow B\|_M^* = 1$, which is due to Lemma 4 iff, for each $A \Rightarrow B \in T$, $M \in \text{Mod}^\bullet(\{A \Rightarrow B\}^*)$, which is iff $M \in \text{Mod}^\bullet(T^*)$.

(ii) is a consequence of (i) and previous observations on properties of models of fuzzy attribute implications [11]. Indeed, (i) ensures that $cl_T^*(A)$ (the least model of T under * generated by A) equals $cl_{T^*}^\bullet(A)$ (the least model of T^* under $^\bullet$ generated by A). Thus, for each $A \Rightarrow B \in T$, we get: $\|A \Rightarrow B\|_T^* = S(B, cl_T^*(A)) = S(B, cl_{T^*}^\bullet(A))$, showing (ii). Claim (iii) is a particularization of (ii). $\blacksquare$

Remark 6: For readers familiar with [13]: Theorem 5 can also be proved using recent results on syntactic entailment of fuzzy attribute implications. Namely, in [13] it has been shown that $T \vdash A \Rightarrow B$ (i.e., $A \Rightarrow B$ is provable from T) iff $A \Rightarrow B$ is provable from T^* using only (Ax) and (Cut), i.e. without using (Mul). Due to the completeness of FAL, we get that $\|A \Rightarrow B\|_T^* = 1$ ($A \Rightarrow B$ is fully entailed by T) iff $A \Rightarrow B$ is provable from T^* using only (Ax) and (Cut), i.e. without using (Mul). Furthermore, the latter claim can be restated as follows: $\|A \Rightarrow B\|_T^* = 1$ iff $\|A \Rightarrow B\|_{T^*}^\bullet = 1$ ($A \Rightarrow B$ is fully entailed by T^* under $^\bullet$). From this we are able to prove (i)–(iii) of Theorem 5. We postpone details of the alternative proof to a full version of this paper.

Theorem 5 allows us to check redundancy of $A \Rightarrow B$ in T as follows. We first determine $(T - \{A \Rightarrow B\})^*$ and then we use GLinClosure [12] to compute $cl_{(T-\{A\Rightarrow B\})^*}^\bullet(A)$. Then, $A \Rightarrow B$ is redundant in T under * iff

$$B \subseteq cl_{(T-\{A\Rightarrow B\})^*}^\bullet(A).$$

The computation of $(T - \{A \Rightarrow B\})^*$ is straightforward. In several important cases we can optimize the computation of this set of FAIs by skipping $c\otimes A \Rightarrow c\otimes B$ which are known to be violating $c\otimes B \nsubseteq c\otimes A$. In a more detail, if $\mathbf{L}$ (our structure of truth degrees) is divisible, then from the fact that

Algorithm 9 (Generate Non-redundant Basis):
Input: data table $\langle X, Y, I\rangle$ with fuzzy attributes
Output: non-redundant basis of $\langle X, Y, I\rangle$

  1 **if** $\emptyset^{\Downarrow*\Uparrow} \neq \emptyset$:
  2     $T := \{\emptyset \Rightarrow \emptyset^{\Downarrow*\Uparrow}\}$
  3 **else**:
  4     $T := \emptyset$
  5 $P := \emptyset$
  6 **while** $P \neq Y$:
  7     $P := \text{NEXTCLOSURE}(P, T)$
  8   **if** $P \neq P^{\Downarrow*\Uparrow}$:
  9   **add** $P \Rightarrow P^{\Downarrow*\Uparrow}$ **to** T
 10 **for each** $A \Rightarrow B \in T$:
 11     $T' := (T - \{A \Rightarrow B\})^*$
 12     $M := \text{GLINCLOSURE}(T', A)$
 13   **if** $B \subseteq M$:
 14   **remove** $A \Rightarrow B$ **from** T
 15 **return** T

$c \otimes A \Rightarrow c \otimes B$ violates $c \otimes B \not\subseteq c \otimes A$ we can conclude that all $d \otimes A \Rightarrow d \otimes B$ where $d \leq c$ will violate $d \otimes B \not\subseteq d \otimes A$. Indeed, this claim is justified by the following assertions.

Lemma 7: *Let* $\mathbf{L}$ *be divisible and let* $c \otimes a \leq c \otimes b$. *Then, for each* $d \leq c$, *we have* $d \otimes a \leq d \otimes b$.

Proof: Take $d \in L$ such that $d \leq c$. Since $\mathbf{L}$ is a divisible residuated lattice, we have $d = c \wedge d = c \otimes (c \rightarrow d)$. Using $c \otimes a \leq c \otimes b$, it follows that

$$d \otimes a = (c \otimes (c \rightarrow d)) \otimes a = (c \rightarrow d) \otimes (c \otimes a) \leq$$
$$\leq (c \rightarrow d) \otimes (c \otimes b) = (c \otimes (c \rightarrow d)) \otimes b = d \otimes b,$$

which is the desired inequality. ∎

Theorem 8: *Let* $\mathbf{L}$ *be divisible residuated lattice,* $A, B \in \mathbf{L}^Y$, *and* $c \in L$ *such that* $c \otimes B \subseteq c \otimes A$. *Then, for each* $d \in L$ *such that* $d \leq c$, *we get* $d \otimes B \subseteq d \otimes A$.

Proof: The assertion can be proved by a componentwise application of Lemma 7. ∎

Thus, if we have a divisible structure of truth degrees, which is quite common in applications (e.g., complete residuated lattices defined on the real unit interval using continuous t-norms are divisible), we can omit FAIs which would violate the above-mentioned condition provided we go through the fixed points of * in a descending order. Note that Theorem 8 cannot be extended to arbitrary (complete) residuated lattices, i.e. in general it is not possible to skip FAIs $d \otimes A \Rightarrow d \otimes B$ where $d \leq c$ if $c \otimes B \not\subseteq c \otimes A$ is violated.

Algorithm 9 summarizes our method of finding a non-redundant basis of $\langle X, Y, I\rangle$. This algorithm encompasses two stages: first, the computation of a complete set given by Theorem 3 (lines 1–9) and then the reduction described in this section (lines 10–14). $\text{NEXTCLOSURE}(P, T)$ denotes the lexically smallest successor of P which either is a fixed point of $^{\Downarrow*\Uparrow}$ or it is in $\mathcal{P}$, see [11] for details. The successor is computed with a polynomial time delay,

see [3], [4] for a more detailed description. Moreover, by $\text{GLINCLOSURE}(T', A)$ we denote the least model of T' (under globalization) containing A. Algorithm GLINCLOSURE computes the least model with asymptotic complexity $O(n)$, where n is the size of T' (the sum of lengths of antecedents of FAIs from T'), see [12].

Proof of soundness of Algorithm 9: Computation of the complete set (lines 1–9) is sound due to our previous observations, see [3], [4]. Soundness of the reduction (lines 10–14) follows from Theorem 5 and from the fact that at the end of the loop (line 15), T cannot contain a FAI which is redundant in T under *. By contradiction, if there were such a FAI $A \Rightarrow B \in T$, then $A \Rightarrow B$ would be true in each model of $T - \{A \Rightarrow B\}$ which contradicts the fact that $A \Rightarrow B$ were not true in model M (line 12) of a superset of $T - \{A \Rightarrow B\}$ when $A \Rightarrow B$ was processed in the body of the loop (lines 11–14). The rest is obvious. ∎

By inspecting lines 10–14 of Algorithm 9, we can see that the asymptotic complexity of reduction is $O(pn)$, where $n = |T|$ (we go through n FAIs in T which are computed in the first stage) and $p = \sum_{A \Rightarrow B \in T^*} |\{y \in Y \mid A(y) > 0\}|$ (computation of each least model invoked at line 12 takes at most p steps). Note that the test of redundancy in Algorithm 9 is more efficient than the initial variant in [11] which used more iterations (through formulas from T) than it is actually necessary.

IV. ILLUSTRATIVE EXAMPLES AND EXPERIMENTS

In this section we present examples of non-redundant bases and experimental observations concerning the algorithm presented in the previous section. In the sequel, $\mathbf{L}$ is always assumed to be a five-element Łukasiewicz chain, see [2].

Example 10: Consider again the data table from Fig. 1. If we take a globalization as the hedge, the non-redundant basis produced by Algorithm 9 is the following:

$\{\{^{.75}/G\} \Rightarrow \{G\}, \{^{.25}/P, G\} \Rightarrow \{A, P, G\}, \{^{.5}/P\} \Rightarrow \{^{.5}/P, ^{.25}/G\},$
$\{^{.75}/P, ^{.25}/G\} \Rightarrow \{^{.5}/A, ^{.75}/P, ^{.5}/G\}, \{^{.25}/A\} \Rightarrow \{^{.25}/A, ^{.25}/G\},$
$\{^{.25}/A, G\} \Rightarrow \{A, P, G\}, \{^{.25}/A, ^{.5}/P, ^{.5}/G\} \Rightarrow \{^{.5}/A, ^{.75}/P, ^{.5}/G\},$
$\{^{.5}/A, ^{.25}/G\} \Rightarrow \{^{.5}/A, ^{.5}/G\}, \{^{.5}/A, ^{.25}/P, ^{.5}/G\} \Rightarrow \{^{.5}/A, ^{.75}/P, ^{.5}/G\},$
$\{^{.5}/A, P, ^{.5}/G\} \Rightarrow \{^{.75}/A, P, ^{.5}/G\}, \{^{.75}/A, ^{.5}/G\} \Rightarrow \{^{.75}/A, ^{.75}/P, ^{.5}/G\},$
$\{A, P, ^{.5}/G\} \Rightarrow \{A, P, G\}\}.$

Note that in case of globalization, lines 10–14 of the algorithm do not have any effect, because the set T produced in the first stage of the algorithm is already non-redundant. If we take the identity instead of the globalization, the non-redundant basis will be smaller:

$\{\{^{.75}/P, ^{.25}/G\} \Rightarrow \{^{.25}/A, ^{.75}/P, ^{.5}/G\}, \{^{.25}/A, ^{.5}/P, G\} \Rightarrow \{^{.5}/A, ^{.5}/P, G\},$
$\{^{.25}/A, P, ^{.5}/G\} \Rightarrow \{^{.5}/A, P, ^{.5}/G\}, \{^{.5}/A, ^{.25}/G\} \Rightarrow \{^{.5}/A, ^{.5}/G\},$
$\{^{.5}/A, ^{.25}/P, G\} \Rightarrow \{^{.5}/A, ^{.5}/P, G\}, \{A, ^{.25}/P, ^{.5}/G\} \Rightarrow \{A, ^{.5}/P, ^{.5}/G\},$
$\{A, P, ^{.5}/G\} \Rightarrow \{A, P, ^{.75}/G\}\}.$

Example 11: We have run several experiments with randomly generated data tables to explore the relationship

hedge	0	0.25	0.5	0.75	1
$\bullet = *_1$	0	0	0	0	1
$*_2$	0	0	0.5	0.5	1
$*_3$	0	0.25	0.25	0.25	1
$*_4$	0	0.25	0.5	0.5	1
$*_5$	0	0.25	0.5	0.75	1

truth degrees in data					average size of non-redundant basis				
0	0.25	0.5	0.75	1	$\bullet$	$*_2$	$*_3$	$*_4$	$*_5$
20%	20%	20%	20%	20%	99	84	92	92	84
non-redundant implications ratio					100%	69%	94%	80%	51%
6.25%	25%	37.5%	25%	6.25%	71	60	68	61	55
non-redundant implications ratio					100%	76%	97%	83%	52%
36.9%	36.9%	18.5%	6.2%	1.5%	66	61	68	83	53
non-redundant implications ratio					100%	56%	84%	71%	33%
23%	19%	16%	19%	23%	101	89	94	97	86
non-redundant implications ratio					100%	70%	94%	81%	50%

Fig. 2. Definitions of all hedges on a five-element Łukasiewicz chain (left); average sizes of bases according to distributions of truth degrees (right)
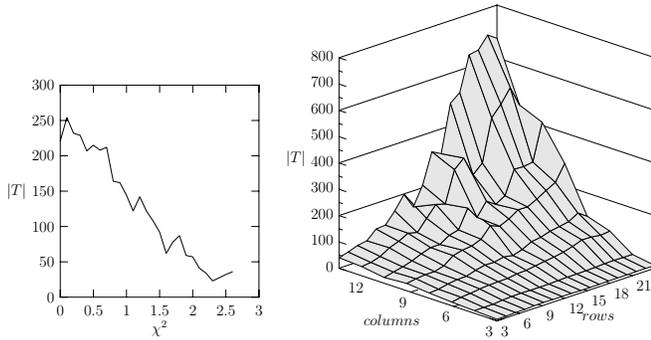

Fig. 3. Maximal sizes of bases depending on the structure of data tables

between complete sets of FAI computed in the first stage of the algorithm (lines 1–9) and the non-redundant bases which result by reducing the initial complete sets (lines 10–14). The behavior of the algorithm was explored for data tables with a fixed size but with various distributions of truth degrees present in the tables because previous experiments indicated that distributions of truth degrees in data tables seem to be correlating with the sizes of bases. In Fig. 2 (left) we have definitions of all possible hedges on **L**. The table in Fig. 2 (right) summarizes average sizes of non-redundant bases for each of the hedges with respect to four different distributions of truth degrees in data tables: uniform, binomial, Poisson, and beta. The ratios (in %) describe how many of the FAIs from the initial complete sets are non-redundant. In case of globalization, the ratio is always 100% but in some cases it can be significantly smaller. During our experiments with data of various sizes, we have found an interesting correspondence between the uniformity of the distribution of truth degrees and sizes of bases. Roughly speaking, the more uniform the distribution of truth degrees in a data table is, the more FAIs its non-redundant basis has. The situation is depicted in Fig. 3 (left), where the x-axis contains values of $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$ (a part of Pearson's χ^2-test), where for O_i we take frequencies of truth degrees in data and E_i represent the desired frequency of truth degree in the uniform distribution (both normalized with respect to table size).

Example 12: We have also studied the maximal number of implications in non-redundant basis (produced by previous algorithm) for different sizes of input tables. The maximal sizes of bases are depicted in the 3D graph in Fig. 3 (right).

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