Compositions of Fuzzy Relations With Hedges

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***Abstract*— We present extensions of ordinary compositions of fuzzy relations. The extensions consist in parameterizing the ordinary compositions by means of particular unary functions on the scales of truth degrees. The approach is inspired by our previous work on formal concept analysis of data with fuzzy attributes where such parameterization of one particular type of fuzzy relational composition was used to control the number of clusters extracted from data. We present definitions and basic properties of the parameterized compositions, examples, and implications for several domains of application of fuzzy relational compositions.**

I. INTRODUCTION

Fuzzy relations are the basic tool in most of the applications of fuzzy logic. The concept of a fuzzy relation generalizes the concept of an ordinary relation. Likewise, the calculus of fuzzy relations generalizes the calculus of ordinary relations. Among the important operations with fuzzy relations are various types of composition of fuzzy relations. In our paper, we will focus on so-called $\circ$-composition, $\triangleleft$-composition, and $\triangleright$-composition. These compositions were studied by Bandler and Kohout, see [2], [3], and also [4].

We propose and investigate a simple extension of the above-mentioned compositions which results by inserting a particular unary function on the set of truth degrees in the standard definition of compositions. The functions we consider are truth functions of logical connectives "very true", called truth-stressing hedges, which were studied in fuzzy logic in narrow sense. Particular cases of our parameterized composition of fuzzy relations were used in the investigation to formal concept analysis of data with fuzzy attributes [8]. Two lessons can be taken from this investigation. First, extension by hedges yields a general framework which leaves previous attempts as particular cases. Therefore, one gets a single theory which covers several particular instances. Second, extension by hedges yields a parameterized approach. In the particular case of formal concept analysis, the role of a hedge is to control the size of extracted clusters from data.

This paper attempts to generalize our previous experience and to introduce compositions of fuzzy relations parameterized by hedges. After introducing preliminaries in Section II, we present definitions and basic properties of compositions with hedges in Section III. In Section IV we present applications of our approach. Section V presents topics for future research.

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II. PRELIMINARIES

In this section we present an overview of notions of fuzzy logic and fuzzy set theory we will be using in this paper. Details can be found e.g. in [4], [12], [14], a good introduction to fuzzy logic and fuzzy sets is presented in [16].

A. Complete Residuated Lattices

Our approach to fuzzy sets and fuzzy relations is based on complete residuated lattices which are used as basic structures of truth degrees. A complete residuated lattice [4], [14] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. That fact that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice means that general infima $\bigwedge_{i\in I} a_i$ and general suprema $\bigvee_{i\in I} a_i$ exist for any subset $\{a_i \,|\, i \in I\} \subseteq L$. As it is usual in the context of fuzzy logic, elements $a \in L$ are called truth degrees. Operations $\otimes$ and $\rightarrow$ are truth functions of logical connectives "fuzzy conjunction" (also called "multiplication") and "fuzzy implication" (also called "residuum"). For each complete residuated lattice $\mathbf{L}$, we consider a derived binary operation $\leftrightarrow$ ("fuzzy equivalence/biconditional" also called "biresiduum") defined, for each $a, b \in L$, by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. We denote by $\leq$ the lattice order induced by $\mathbf{L}$. Using the adjointness property $a \leq b$ iff $a \rightarrow b = 1$. Complete residuated lattice $\mathbf{L}$ is called linearly ordered (or a chain) if, for each $a, b \in L$, $a \leq b$ or $b \leq a$.

The most important complete residuated lattices are those defined on the real unit interval. In such a case, $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz: $a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$; Gödel (minimum): $a \otimes b = a \wedge b$, $a \rightarrow b = b$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$; Goguen (product): $a \otimes b = a \cdot b$, $a \rightarrow b = \frac{b}{a}$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$.

A special case of a complete residuated lattice is the two-element Boolean algebra $\langle \{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic.

Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

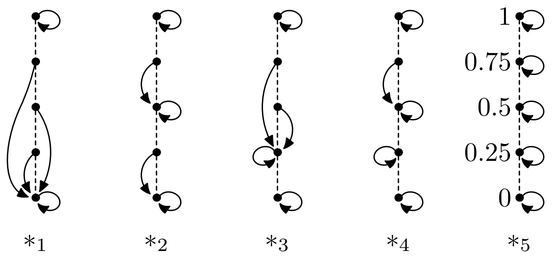

Fig. 1. Hedges on a five-element Łukasiewicz chain

B. Truth-Stressing Hedges

Complete residuated lattices can be equipped with additional fundamental operations. In this paper, we are going to use particular unary operations called truth-stressing hedges. An idempotent truth-stressing hedge (shortly, a hedge) on a complete residuated lattice $\mathbf{L}$ is a mapping ${}^*: L \to L$ satisfying the following conditions

$$1^* = 1, \tag{1}$$

$$a^* \leq a, \tag{2}$$

$$(a \to b)^* \leq a^* \to b^*, \tag{3}$$

$$a^{**} = a^*, \tag{4}$$

for each $a, b \in L$. Truth-stressing hedges were investigated from the point of view of fuzzy logic in narrow sense by Hájek [15] who showed complete axiomatizations of BL-logics equipped with unary connectives "very true". Our approach to hedges is close to that in [15] although the conditions postulated in [15] are a bit different. Unlike [15], we use the additional condition of idempotency (4). On the other hand, we do not use

$$(a \vee b)^* \leq a^* \vee b^* \tag{5}$$

which is required in (4) for the sake of axiomatization. In order to have desirable properties of compositions of fuzzy relations with hedges, we will postulate further requirements. One of them will be a stronger version on Hájek's (5).

Let us note that properties (2)–(4) have natural interpretations. For instance, (2), called subdiagonality, can be read: "if a is very true, then a is true"; (3) can be read: "if $a \to b$ is very true and if a is very true, then b is very true", etc.

On each complete residuated lattice $\mathbf{L}$, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a$ ($a \in L$);
(ii) globalization [17], i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Note that globalization agrees with Baaz's Δ-operation [1] in case of linearly ordered residuated lattices.

Example 1: Let $\mathbf{L}$ be a finite residuated lattice with $L = \{0, 0.25, 0.5, 0.75, 1\}$, $\wedge$ and $\vee$ being minimum and maximum, respectively, $\otimes$ be the Łukasiewicz conjunction on L with its residuum $\to$. There are five idempotent truth-stressing hedges on $\mathbf{L}$. The hedges are depicted by their diagrams in Fig. 1. The left-most hedge $*_1$ is globalization, see (6). On the contrary, the right-most hedge $*_5$ is identity. There are three intermediate hedges $*_2$, $*_3$, and $*_4$. For instance, in case of $*_2$, we have $0^{*_2} = 0.25^{*_2} = 0$, $0.5^{*_2} = 0.75^{*_2} = 0.5$, and $1^{*_2} = 1$.

Example 2: There is just one hedge on the two-element Boolean algebra $\mathbf{2}$. Namely, hedge * such that $0^* = 0$ and $1^* = 1$. Hence, in case of the two-element Boolean algebra, globalization coincides with identity.

C. Fuzzy Sets and Fuzzy Relations

Suppose that $\mathbf{L}$ is our structure of truth degrees. We define usual notions: a fuzzy set (an $\mathbf{L}$-set) A in universe U is a mapping $A: U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Operations with fuzzy sets are defined componentwise. For instance, the intersection of fuzzy sets $A, B \in \mathbf{L}^U$ is a fuzzy set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. A binary fuzzy relation (a binary $\mathbf{L}$-relation) I between X and Y is a mapping $I\colon X \times Y \to L$, $I(x, y)$ being interpreted as "the degree to which x and y are related by I". By definition, a binary fuzzy relation $I\colon X \times Y \to L$ is a fuzzy set in the universe $X \times Y$, i.e. $I \in \mathbf{L}^{X \times Y}$.

In the following we use well-known properties of residuated lattices, fuzzy sets, and fuzzy relations which can be found in monographs [4], [12], [14], [16].

III. Compositions of Fuzzy Relations with Hedges

Throughout this section, we assume that R and S are $\mathbf{L}$-relations between X and Y, and Y and Z, respectively. Let us recall the definition of $\circ$-composition, $\triangleleft$-composition, and $\triangleright$-composition of fuzzy relations, see e.g. [2], [3], [4].

$$(R \circ S)(x, z) = \textstyle\bigvee_{y \in Y} \big(R(x, y) \otimes S(y, z)\big), \tag{7}$$

$$(R \triangleleft S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(R(x, y) \to S(y, z)\big), \tag{8}$$

$$(R \triangleright S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(S(y, z) \to R(x, y)\big), \tag{9}$$

for all $x \in X$, $z \in Z$. $(R \circ S)(x, z)$ is the truth degree of proposition "there is $y \in Y$ such that $\langle x, y\rangle$ is in R and $\langle y, z\rangle$ is in S". $(R \triangleleft S)(x, z)$ is the truth degree of proposition "for every $y \in Y$: if $\langle x, y\rangle$ is in R then $\langle y, z\rangle$ is in S". $(R \triangleright S)(x, z)$ is the truth degree of proposition "for every $y \in Y$: if $\langle y, z\rangle$ is in S then $\langle x, y\rangle$ is in R". The following definition presents our extension.

Definition 1: Let R and S be $\mathbf{L}$-fuzzy relations between X and Y and between Y and Z. Fuzzy relations $(R \circ^* S)$, $(R \triangleleft^* S)$, and $(R \triangleright^* S)$ between X and Z are defined by

$$(R \circ^* S)(x, z) = \textstyle\bigvee_{y \in Y} \big(R(x, y)^* \otimes S(y, z)\big), \tag{10}$$

$$(R \triangleleft^* S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(R(x, y)^* \to S(y, z)\big), \tag{11}$$

$$(R \triangleright^* S)(x, z) = \textstyle\bigwedge_{y \in Y} \big(S(y, z)^* \to R(x, y)\big), \tag{12}$$

for all $x \in X$, $z \in Z$.

Remark 1: (1) Obviously, one could insert the hedge * at other places, too, in the definition of fuzzy relational compositions. Our motivation for inserting the hedges in the above definition comes from the fact that we use such compositions in examples below.

(2) $(R \circ^* S)(x,z)$ is the truth degree of proposition "there is $y \in Y$ such that it is very true that $\langle x,y\rangle$ is in R and $\langle y,z\rangle$ is in S". $(R \lhd^* S)(x,z)$ is the truth degree of proposition "for every $y \in Y$: if it is very true that $\langle x,y\rangle$ is in R then $\langle y,z\rangle$ is in S". $(R \rhd S)(x,z)$ is the truth degree of proposition "for every $y \in Y$: if it is very true that $\langle y,z\rangle$ is in S then $\langle x,y\rangle$ is in R".

(3) Obviously, if * is the identity, (10), (11), and (12) become (7), (8), and (9). In this sense, the new compositions generalize the ordinary ones.

Let for hedges $*_1$ and $*_2$ put

$$*_1 \leq *_2 \quad \text{iff} \quad \text{for each } a \in L\colon a^{*_1} \leq a^{*_2}.$$

That is, $*_1 \leq *_2$ means that $*_1$ is stronger in that it pushes truth degrees more toward 0. Clearly, if $*_g$ and $*_i$ denote globalization and identity, then $*_g \leq * \leq *_i$ for every hedge $*$. We have the following proposition:

Theorem 1: Let $*_1 \leq *_2$. Then

$$R \circ^{*_1} S \subseteq R \circ^{*_2} S,$$
$$R \lhd^{*_1} S \supseteq R \lhd^{*_2} S,$$
$$R \rhd^{*_1} S \supseteq R \rhd^{*_2} S.$$

Proof: From monotony of $\otimes$

$$\begin{aligned}(R \circ^{*_1} S)(x,z) &= \textstyle\bigvee_{y\in Y}(R^{*_1}(x,y) \otimes S(y,z)) \leq \\ &\leq \textstyle\bigvee_{y\in Y}(R^{*_2}(x,y) \otimes S(y,z)) = \\ &= (R \circ^{*_2} S)(x,z).\end{aligned}$$

From antitony of $\to$ in the first argument

$$\begin{aligned}(R \lhd^{*_1} S)(x,z) &= \textstyle\bigwedge_{y\in Y}(R^{*_1}(x,y) \to S(y,z)) \geq \\ &\geq \textstyle\bigwedge_{y\in Y}(R^{*_2}(x,y) \to S(y,z)) = \\ &= (R \lhd^{*_2} S)(x,z).\end{aligned}$$

The proof of the last inequality is similar. ■

As a consequence, we get:

$$R \circ^* S \subseteq R \circ S,$$
$$R \lhd^* S \supseteq R \lhd S,$$
$$R \rhd^* S \supseteq R \rhd S.$$

Now, we present several properties of the compositions. In some cases, we need additional properties of hedges. In particular, we will use:

$$(a^* \otimes b)^* = a^* \otimes b^*, \tag{13}$$
$$\left(\textstyle\bigvee_{i\in I} a_i\right)^* = \textstyle\bigvee_{i\in I} a_i^*, \tag{14}$$
$$\left(\textstyle\bigwedge_{i\in I} a_i\right)^* = \textstyle\bigwedge_{i\in I} a_i^*. \tag{15}$$

For instance, (13) is satisfied when $*$ is identity or if $*$ satisfies $a^* \otimes a^* = a^*$. (14) holds true in any finite chain or, more generally, in any Noetherian chain [9], i.e. a linearly ordered structure of truth degrees in which every subset has a largest element $(\bigvee_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$; dually for (15).

In the following theorems we will use the corresponding properties of ordinary products and a simple observation. Namely, it is obvious from the definitions that

$$R \circ^* S = R^* \circ S,$$
$$R \lhd^* S = R^* \lhd S,$$
$$R \rhd^* S = R \rhd S^*.$$

Theorem 2 (products and associativity): Let $*$ satisfy (13) and (14). Then

$$R \circ^*(S \circ^* T) = (R \circ^* S) \circ^* T,$$
$$R \lhd^*(S \rhd^* T) = (R \lhd^* S) \rhd^* T,$$
$$R \lhd^*(S \lhd^* T) = (R \circ^* S) \lhd^* T,$$
$$(R \rhd^* S) \rhd^* T = R \rhd^*(S \circ^* T).$$

Proof: Using associativity of $\circ$ and properties (13) and (14), we get

$$\begin{aligned}R \circ^*(S \circ^* T) &= R^* \circ (S^* \circ T) = (R^* \circ S^*) \circ T = \\ &= (R \circ^* S)^* \circ T = (R \circ^* S) \circ^* T.\end{aligned}$$

Likewise, using $R \lhd (S \rhd T) = (R \lhd S) \rhd T$, we get

$$\begin{aligned}R \lhd^*(S \rhd^* T) &= R^* \lhd (S \rhd T^*) = (R^* \lhd S) \rhd T^* = \\ &= (R \lhd^* S) \rhd^* T,\end{aligned}$$

and using $R \lhd (S \lhd T) = (R \circ S) \lhd T$, we get

$$\begin{aligned}R \lhd^*(S \lhd^* T) &= R^* \lhd (S^* \lhd T) = (R^* \circ S^*) \lhd T = \\ &= (R^* \circ S)^* \lhd T = (R \circ^* S) \lhd^* T.\end{aligned}$$

The fourth equality is symmetrical to the third one. ■

Theorem 3 (products and distributivity): Let $*$ satisfy (13) and (14). Then

$$\begin{aligned}
&\left(\textstyle\bigcap_i R_i\right) \circ^* S \subseteq \textstyle\bigcap_i (R_i \circ^* S), && R \circ^* \left(\textstyle\bigcap_i S_i\right) \subseteq \textstyle\bigcap_i (R \circ^* S_i),\\
&\left(\textstyle\bigcup_i R_i\right) \circ^* S = \textstyle\bigcup_i (R_i \circ^* S), && R \circ^* \left(\textstyle\bigcup_i S_i\right) = \textstyle\bigcup_i (R \circ^* S_i),\\
&\textstyle\bigcup_i (R_i \lhd^* S) \subseteq \left(\textstyle\bigcap_i R_i\right) \lhd^* S, && R \lhd^* \left(\textstyle\bigcap_i S_i\right) = \textstyle\bigcap_i (R \lhd^* S_i),\\
&\left(\textstyle\bigcup_i R_i\right) \lhd^* S = \textstyle\bigcap_i (R_i \lhd^* S), && \textstyle\bigcup_i (R \lhd^* S_i) \subseteq R \lhd^* \left(\textstyle\bigcup_i S_i\right),\\
&\left(\textstyle\bigcap_i R_i\right) \rhd^* S = \textstyle\bigcup_i (R_i \rhd^* S), && \textstyle\bigcup_i (R \rhd^* S_i) \subseteq R \rhd^* \left(\textstyle\bigcap_i S_i\right),\\
&\textstyle\bigcap_i (R_i \rhd^* S) \subseteq \left(\textstyle\bigcup_i R_i\right) \rhd^* S, && R \rhd^* \left(\textstyle\bigcup_i S_i\right) = \textstyle\bigcap_i (R \rhd^* S_i).
\end{aligned}$$

Proof: All assertions can be proved by using the distributivity in classical case. For example

$$\begin{aligned}\left(\textstyle\bigcup_i R_i\right) \lhd^* S &= \left(\textstyle\bigcup_i R_i\right)^* \lhd S = \left(\textstyle\bigcup_i R_i^*\right) \lhd S = \\ &= \textstyle\bigcap_i (R_i^* \lhd S) = \textstyle\bigcap_i (R_i \lhd^* S).\end{aligned}$$

■

Theorem 4 (products and inverse relations): Let $*$ satisfy (13) and (14). Then

$$\big(R \circ^{*} S^{*}\big)^{-1} = \big(S^{-1} \circ^{*} R^{-1}\big)^{*},$$
$$\big(R \triangleleft^{*} S\big)^{-1} = S^{-1} \triangleright^{*} R^{-1},$$
$$\big(R \triangleright^{*} S\big)^{-1} = S^{-1} \triangleleft^{*} R^{-1}.$$

Proof: All assertions can be proved by using the properties of inverse relations in classical case. For instance, in case of the the first one, we have

$$\begin{aligned}\big(R \circ^{*} S^{*}\big)^{-1} &= \big(R^{*} \circ S^{*}\big)^{-1} = \big(S^{*}\big)^{-1} \circ \big(R^{*}\big)^{-1} = \\ &= \big(S^{-1}\big)^{*} \circ \big(R^{-1}\big)^{*} = \big((S^{-1})^{*} \circ R^{-1}\big)^{*} = \\ &= \big(S^{-1} \circ^{*} R^{-1}\big)^{*}.\end{aligned}$$

■

For products and their behavior w.r.t. a fuzzy equality (similarity) $\approx$ defined by

$$P_1 \approx P_2 = \textstyle\bigwedge_{\langle u,v\rangle \in U\times V}\big(P_1(u,v) \leftrightarrow P_2(u,v)\big),$$

we need the following lemma.

Lemma 5: Let $v = \bigwedge_{a\in L}(a \rightarrow a^{*})$, then

$$v \otimes (b \rightarrow c) \leq b^{*} \rightarrow c^{*}$$

for all $b, c \in L$.

Proof: $\rightarrow$ is antitone in the first argument, and $\otimes$ is monotone in both of its arguments. Thus, we can write

$$v \otimes (b \rightarrow c) \leq v \otimes (b^{*} \rightarrow c).$$

Therefore, we need to prove $v \otimes (b^{*} \rightarrow c) \leq b^{*} \rightarrow c^{*}$. By adjointness property we get $v \leq (b^{*} \rightarrow c) \rightarrow (b^{*} \rightarrow c^{*})$, which is true because $c \rightarrow c^{*} \leq (b^{*} \rightarrow c) \rightarrow (b^{*} \rightarrow c^{*})$ and $v = \bigwedge_{a\in L}(a \rightarrow a^{*}) \leq c \rightarrow c^{*}$. ■

Theorem 6 (products and similarity):
Let $v = \bigwedge_{a\in L}(a \rightarrow a^{*})$ and $*$ satisfy (13) and (14). Then

$$v \otimes (R_1 \approx R_2) \otimes (S_1 \approx S_2) \leq (R_1 \circ^{*} S_1) \approx (R_2 \circ^{*} S_2),$$
$$v \otimes (R_1 \approx R_2) \otimes (S_1 \approx S_2) \leq (R_1 \triangleleft^{*} S_1) \approx (R_2 \triangleleft^{*} S_2),$$
$$v \otimes (R_1 \approx R_2) \otimes (S_1 \approx S_2) \leq (R_1 \triangleright^{*} S_1) \approx (R_2 \triangleright^{*} S_2).$$

Proof: First, we prove that

$$v \otimes (R_1 \approx R_2) \leq R_1^{*} \approx R_2^{*}.$$

Using Lemma 5, we can write

$$\begin{aligned}v \otimes (R_1 \approx R_2) &\leq \textstyle\bigwedge_{x,y} v \otimes (R_1(x,y) \leftrightarrow R_2(x,y)) \leq \\ &\leq \textstyle\bigwedge_{x,y}(R_1(x,y)^{*} \leftrightarrow R_2(x,y)^{*}) = \\ &= R_1^{*} \approx R_2^{*}.\end{aligned}$$

Using the previous inequality,

$$\begin{aligned}v \otimes (R_1 \approx R_2) \otimes (S_1 \approx S_2) &\leq (R_1^{*} \approx R_2^{*}) \otimes (S_1 \approx S_2) \leq \\ &\leq (R_1^{*} \circ S_1) \approx (R_2^{*} \circ S_2) = \\ &= (R_1 \circ^{*} S_1) \approx (R_2 \circ^{*} S_2).\end{aligned}$$

Proofs of the remaining claims are similar. ■

	v	w	x	y	z
p_1	0.25	0.50	0.50	0.00	0.25
p_2	1.00	0.75	1.00	1.00	0.75
p_3	0.75	1.00	0.75	1.00	0.50
p_4	0.50	0.75	0.00	0.75	0.00

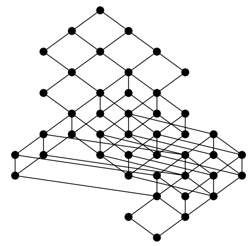

Fig. 2. Data table with fuzzy attributes and $\mathcal{B}(X, Y, I)$

IV. Applications of Hedge-Based Compositions

A. Concept Lattices With Hedges

Concept lattices [10] are lattice-ordered hierarchies of conceptual clusters hidden in object-attribute data. Concept lattices with hedges [6], [8] are, roughly speaking, particular concept lattices, where the hedges serve as parameters controlling the number of clusters extracted from data with graded attributes. In this section, we recall basic notions of concept lattices with hedges and show that the concept-forming operators [5], [6], [8] are particular cases of composition of fuzzy relations with hedges.

Let X and Y be sets of objects and attributes, respectively, I be a binary fuzzy relation between X and Y with $I(x,y)$ being interpreted as a degree to which object $x \in X$ has attribute $y \in Y$. The triplet $\langle X, Y, I\rangle$ is called a data table with fuzzy attributes and can be seen as a table with rows corresponding to objects from X, columns corresponding to attributes from Y, and table entries being truth degrees $I(x,y)$, see Fig. 2 (left) for example.

Let $*_X$ and $*_Y$ be hedges on $\mathbf{L}$. For fuzzy sets $A \in \mathbf{L}^X$ (fuzzy set of objects), $B \in \mathbf{L}^Y$ (fuzzy set of attributes) we define fuzzy sets $A^{\uparrow} \in \mathbf{L}^Y$ (fuzzy set of attributes shared by all objects from A) and $B^{\downarrow} \in \mathbf{L}^X$ (fuzzy set of objects sharing all attributes from B) by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x\in X}\big(A(x)^{*_X} \rightarrow I(x,y)\big), \quad (16)$$
$$B^{\downarrow}(x) = \textstyle\bigwedge_{y\in Y}\big(B(y)^{*_Y} \rightarrow I(x,y)\big). \quad (17)$$

Operators $^{\downarrow}, ^{\uparrow}$ are the concept-forming operators sending fuzzy sets of attributes to fuzzy sets of objects and *vice versa.* Note that $^{\downarrow}, ^{\uparrow}$ form a Galois connection with hedges [6], [8].

With $\langle X, Y, I\rangle$ playing the role of an input data table, put

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \,|\, A^{\uparrow} = B \text{ and } B^{\downarrow} = A\}. \quad (18)$$

Moreover, for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1). \quad (19)$$

$\langle \mathcal{B}(X^{*_X}, Y^{*_Y}, I), \leq\rangle$ is called a (fuzzy) concept lattice with hedges ^{*_X} and ^{*_Y} induced by $\langle X, Y, I\rangle$ [6], [8]. For $*_Y$ being identity, we write only $\mathcal{B}(X^{*_X}, Y, I)$; if both $*_X$ and $*_Y$ are identities, we write $\mathcal{B}(X, Y, I)$, etc.

Elements $\langle A, B\rangle$ of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ are naturally interpreted as concepts (clusters) hidden in the input data represented by I. Namely, $A^{\uparrow} = B$ and $B^{\downarrow} = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all

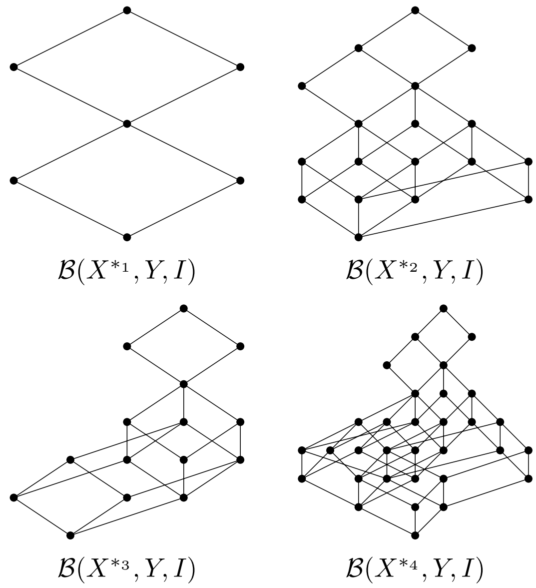

Fig. 3. Concept lattices with hedges

attributes from B. These conditions formalize the definition of a concept as developed in Port-Royal logic; A and B are called the extent and the intent of the concept $\langle A, B\rangle$, respectively, and represent the collection of all objects and all attributes covered by $\langle A, B\rangle$. The ordering $\leq$ defined by (19) models a subconcept-superconcept hierarchy. For details, we refer to [4], [5], [6], [8].

Example 3: Let $X = \{p_1, p_2, p_3, p_4\}$ be a set of objects representing people taking part of a survey. Let $Y = \{v, w, x, y, z\}$ be a set of the following socio-economic attributes: v – high salary; w – high monthly spending; x – high savings; y – expensive insurance; z – high purchasing power. The data table in Fig. 2 (left) defines a binary fuzzy relation $I \in \mathbf{L}^{X\times Y}$ among objects and attributes. Suppose that our structure of truth degrees is the five-element Łukasiewicz algebra. If both $*_X$ and $*_Y$ are identities, the resulting concept lattice has 45 formal concepts. The hierarchy of the concepts is depicted in Fig. 2 (right). Different choices of hedges allow us to have smaller hierarchies of concepts, focusing thus on concepts which are more relevant. For instance, if we let $*_Y$ be identity and let $*_X$ be $*_1, \dots, *_4$ from Fig. 1, we obtain four concept lattices $\mathcal{B}(X^{*_i}, Y, I)$ which are depicted in Fig. 3.

Consider now the operator $^{\uparrow}$ defined by (16). Take a fuzzy set $A \in \mathbf{L}^X$ of objects. Furthermore, take a set $Z = \{z\}$ and define a binary fuzzy relation $R \in \mathbf{L}^{Z\times X}$ by $R(z, x) = A(x)$ ($x \in X$). Note that R can also be seen as a "fuzzy set of objects" to which an object $x \in X$ belongs to a degree $R(z, x)$. Using (11) and (16), we clearly have

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x\in X}\big(A(x)^{*_X} \to I(x, y)\big) =$$
$$= \textstyle\bigwedge_{x\in X}\big(R(z, x)^{*_X} \to I(x, y)\big) = \big(R \lhd^{*} I\big)(z, y).$$

Again, $R \lhd^{*} I \in \mathbf{L}^{Z\times Y}$ is a fuzzy relation representing, in fact, a fuzzy set of attributes. Therefore, $A^{\uparrow}$ can be seen as a result of the $\lhd^{*}$-composition of R and I.

Dually, for any fuzzy set $B \in \mathbf{L}^Y$ of attributes, we can define $S \in \mathbf{L}^{Y\times Z}$ by $S(y, z) = B(y)$ ($y \in Y$). Then, using (12) and (17),

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y\in Y}\big(B(y)^{*_Y} \to I(x, y)\big) =$$
$$= \textstyle\bigwedge_{y\in X}\big(S(y, z)^{*_Y} \to I(x, y)\big) = \big(I \rhd^{*} S\big)(x, z),$$

i.e. $B^{\uparrow}$ can be seen as a result of the $\rhd^{*}$-composition of I and S. Therefore, the concept-forming operators can be seen as a particular case of compositions of fuzzy relations with hedges.

B. Threshold-Based Compositions

The ordinary $\lhd$-product of fuzzy relations can be modified in order to account for a natural requirement of including a threshold in definition of $R \lhd S$. Namely, for threshold $\delta \in L$, one can put

$$(R \lhd_{\delta} S)(x, z) = \textstyle\bigwedge_{y\in Y, R(x,y)\geq\delta} S(y, z).$$

That is, $(R \lhd_{\delta} S)(x, z)$ is a truth degree of "for every y: if x and y are R-related to degree at least δ then y and z are S-related". Alternatively, $(R \lhd_{\delta} S)(x, z)$ is the degree to which z is related to every y for which $R(x, y) \geq \delta$. A similar idea is used in the framework of concept lattices in [18]. Note that if R and S are crisp relations and $\delta = 1$, $R \lhd_{\delta} S$ coincides with the ordinary $\lhd$-composition of ordinary (bivalent) relations. Therefore, like $R \lhd S$, $R \lhd_{\delta} S$ is a generalization of the ordinary $\lhd$-product of (bivalent) relations, but a one which is different from $R \lhd S$.

Now, both $\lhd$ given by (11) and $\lhd_{\delta}$ are definable using $\lhd^{*}$. Namely, as mentioned above, if $*$ is the identity on L, $R \lhd S$ coincides with $R \lhd^{*} S$. Furthermore, one can easily check the following theorem.

Theorem 7: If $*$ is globalization (6), $R \lhd_{\delta} S$ coincides with $(\delta \to R) \lhd^{*} S$.

We stop our visit to threshold-inspired compositions by noting that analogous observations can be made with the other types of relational compositions.

C. Relational Equations With Hedges

We now consider the problem of fuzzy relational equations [13]. The problem can be described as: Given R and T, determine S for which

$$R \odot S \text{ is (at least approximately) equal to } T. \quad (20)$$

Alternatively, given S and T, determine R. We denote by

$$U \odot S = T, \quad R \odot U = T$$

a fuzzy relational equation, where the unknown relation is R and S, respectively. Now, due to limited scope, we present the criteria of solvability of fuzzy relational equations for the particular case where $\odot = \circ^{*}$ and where exact equality is required in (20).

First, we need the following lemma which shows the criteria of solvability of fuzzy relational equations with $\circ$-composition. The result is well-known, see [13] for $L = [0, 1]$ and [4] for the general case.

Lemma 8: A fuzzy relational equation $U \circ S = T$ has a solution iff $(S \triangleleft T^{-1})^{-1}$ is a solution.

Theorem 9: A fuzzy relational equation $R \circ^* U = T$ has a solution iff $(R^*)^{-1} \triangleleft T$ is a solution.

Proof: The result follows directly from Lemma 8. Indeed, using the properties of inverse relations we get $R \circ^* U = T$ is equivalent to $U^{-1} \circ (R^*)^{-1} = T^{-1}$. Therefore, if $R \circ^* U = T$ has solution, Lemma 8 implies that $U^{-1} = \left((R^*)^{-1} \triangleleft T\right)^{-1}$ is a solution of $U^{-1} \circ (R^*)^{-1} = T^{-1}$, i.e. $(R^*)^{-1} \triangleleft T$ is a solution of $R \circ^* U = T$. ■

For fuzzy relational equation $U \circ^* S = T$ we start with the following result which immediately follows from definitions.

Theorem 10: R is a solution of $U \circ^* S = T$ iff R^* is a solution of $U \circ S = T$ iff R^* is a solution of $U \circ^* S = T$.

Theorem 10 shows an interesting fact. Let $*$ be globalization. Then R^* is a crisp relation, i.e. $R^*(x, y) = 0$ or $R^*(x, y) = 1$ for any x, y. Therefore, due to Theorem 10, $U \circ^* S = T$ has a solution if and only if the ordinary fuzzy relational equation $U \circ S = T$ has a solution which is a crisp relation. In general, $U \circ^* S = T$ has a solution if and only if the ordinary fuzzy relational equation $U \circ S = T$ has a solution R for which $R(x, y) \in \{a^* \mid a \in L\}$. Therefore, solvability of $U \circ^* S = T$ can be regarded as solvability of the ordinary $U \circ S = T$ with an additional constraint imposed by the hedge $*$. The following theorem provides criteria of solvability of $U \circ^* S = T$.

Theorem 11: A fuzzy relational equation $U \circ^* S = T$ has a solution iff $(S \triangleleft T^{-1})^{-1}$ is a solution.

Proof: Since $U \circ^* S = U^* \circ S$, we see that if U is a solution then $U^* \leq (S \triangleleft T^{-1})^{-1}$ (use adjointness and standard manipulation). Due to (4), $U^* \leq (S \triangleleft T^{-1})^{-1}$ is equivalent to $U^* \leq \left((S \triangleleft T^{-1})^{-1}\right)^*$. Therefore, $T = U \circ^* S = U^* \circ S \leq \left((S \triangleleft T^{-1})^{-1}\right)^* \circ S \leq (S \triangleleft T^{-1})^{-1} \circ S \leq T$, which shows that $(S \triangleleft T^{-1})^{-1} \circ^* S = T$. ■

Example 4: As an example, let $*$ be globalization and consider Łukasiewicz operations on $L = [0, 1]$. Let S and T be represented by matrices $\begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix}$ and $\begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}$. One can check using Lemma 8 that $U \circ S = T$ has a solution. For example, one such solution is $(S \triangleleft T^{-1})^{-1}$ with the representing matrix being $\begin{pmatrix} 1 & 0.7 \\ 0.9 & 0.5 \end{pmatrix}$, i.e.

$$\begin{pmatrix} 1 & 0.7 \\ 0.9 & 0.5 \end{pmatrix} \circ \begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}.$$

On the other hand, there is no solution of $U \circ^* S = T$, i.e. there is no binary matrix B for which

$$B \circ \begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}.$$

This follows from Theorem 11 by observing that $\left((S \triangleleft T^{-1})^{-1}\right)^*$ whose matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not a solution.

V. Future Research

Future research will include the following topics.

– A detailed investigation of properties of compositions with hedges.
– Decompositions of fuzzy relations where, as shown above, inclusion of hedges practically means imposing a constraint on the unknown fuzzy relation such as requiring that the fuzzy relation is in fact an ordinary (crisp) relation.
– There is an interesting connection of $\circ$-composition with hedges to fuzzy attribute implications [7]. Namely, models of particular theories of the logic fuzzy attribute implications are just fixed points of operators defined by compositions of fuzzy relations with hedges.
– As mentioned above, concept lattices with hedges [6], [8] are based on a $\triangleleft$-composition with hedges. We suppose that for other types of fuzzy concept lattices, see [11], employment of hedges yields a feasible approach in which hedges will play a similar role as in [6], [8].

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