Isotone Galois connections and concept lattices with hedges

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Abstract— **We study isotone fuzzy Galois connections and concept lattices parameterized by truth-stressing hedges. Isotone fuzzy Galois connections and concept lattices provide an alternative to antitone fuzzy Galois connections and concept lattices which are the foundational structures for formal concept analysis of data with fuzzy attributes. We demonstrate that hedges enable us to control the number of fixed points of Galois connections, i.e. collections of objects and attributes which represent interesting clusters in data. In addition, we present properties of isotone connections with hedges, including their axiomatization, and describe the structure of the associated concept lattices.**

Index Terms—**Concept lattice, fuzzy logic, Galois connection, hedge**

I. INTRODUCTION

Fuzzy Galois connections, closure operators, and concept lattices are the basic structures behind formal concept analysis of data with fuzzy attributes, see [3, 4, 6]. In our previous work, we generalized these structures using particular unary functions called truth-stressing hedges [8, 9]. These functions serve as parameters which influence the number of formal concepts, i.e. the clusters extracted from data. In [12, 17], the authors developed an alternative approach to fuzzy Galois connections. Because of natural properties of the ordinary and the alternative connections, they call the ordinary connections antitone and the alternative ones isotone. In classical setting, both the ordinary and the alternative approaches are equivalent. However, in a fuzzy setting, they are not. The aim of this paper is to extend the alternative approach by truth-stressing hedges. Our motivation is to have a parameterized version of isotone Galois connections where the parameters control the number of fixed points, i.e. clusters extracted from data. In this paper, we develop foundations and provide illustrative examples. In particular, Section II provides preliminaries from fuzzy logic and fuzzy sets. In Section III, we study isotone Galois connections with hedges. The associated concept lattices are covered in Section IV. Section V provides illustrative examples. Further issues and conclusions are summarized in Section VI.

II. PRELIMINARIES

In this section we present an overview of notions of fuzzy logic and fuzzy set theory. For details we refer to [5, 14].

A. Complete Residuated Lattices

We use complete residuated lattices as basic structures of truth degrees. A complete residuated lattice [5, 14] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. That fact that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice means that general infima $\bigwedge_{i\in I} a_i$ and general suprema $\bigvee_{i\in I} a_i$ exist for any subset $\{a_i \,|\, i \in I\} \subseteq L$. As it is usual in the context of fuzzy logic, elements $a \in L$ are called truth degrees. Operations $\otimes$ and $\rightarrow$ are truth functions of logical connectives "fuzzy conjunction" (also called "multiplication") and "fuzzy implication" (also called "residuum"). For each complete residuated lattice $\mathbf{L}$, we consider a derived binary operation $\leftrightarrow$ ("fuzzy equivalence/biconditional" also called "biresiduum") defined, for each $a, b \in L$, by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. We denote by $\leq$ the lattice order induced by $\mathbf{L}$. Using the adjointness property $a \leq b$ iff $a \rightarrow b = 1$. Complete residuated lattice $\mathbf{L}$ is called linearly ordered (or a chain) if, for each $a, b \in L$, $a \leq b$ or $b \leq a$.

The most important complete residuated lattices are those defined on the real unit interval. In such a case, $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz: $a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$; Gödel (minimum): $a \otimes b = a \wedge b$, $a \rightarrow b = b$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$; Goguen (product): $a \otimes b = a \cdot b$, $a \rightarrow b = \frac{b}{a}$ for $a > b$ and $a \rightarrow b = 1$ for $a \leq b$.

A special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic.

Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

B. Truth-Stressing Hedges

We will use particular additional unary operations called truth-stressing hedges. An idempotent truth-stressing hedge (shortly, a hedge) on a complete residuated lattice $\mathbf{L}$ is a mapping ${}^*\colon L \rightarrow L$ satisfying the following conditions

$$1^* = 1, \quad (1)$$

$$a^* \leq a, \quad (2)$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \quad (3)$$

$$a^{**} = a^*, \quad (4)$$

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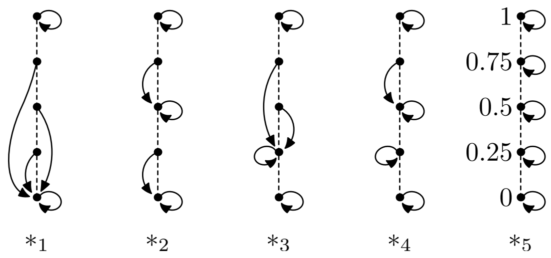

Fig. 1. Hedges on a five-element Łukasiewicz chain

for each $a, b \in L$. Truth-stressing hedges were investigated from the point of view of fuzzy logic in narrow sense by Hájek [15] who showed complete axiomatizations of BL-logics equipped with unary connectives "very true". Our approach to hedges is close to that in [15] although the conditions postulated in [15] are a bit different. Unlike [15], we use the additional condition of idempotency (4). On the other hand, we do not use $(a \vee b)^* \leq a^* \vee b^*$ which is required in [15]. If φ is a formula and $||\varphi||$ is the truth degree of φ then $||\varphi||^*$ is the truth degree of "very φ" ("it is very true that φ").

On every complete residuated lattice $\mathbf{L}$, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a$ ($a \in L$);

(ii) globalization [18], i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Example 1: Let $\mathbf{L}$ be a finite residuated lattice with $L = \{0, 0.25, 0.5, 0.75, 1\}$, $\wedge$ and $\vee$ being minimum and maximum, respectively, $\otimes$ be the Łukasiewicz conjunction on L with its residuum $\rightarrow$. There are five idempotent truth-stressing hedges on $\mathbf{L}$. The hedges are depicted by their diagrams in Fig. 1. The left-most hedge $*_1$ is globalization, see (5). On the contrary, the right-most hedge $*_5$ is identity. There are three intermediate hedges $*_2$, $*_3$, and $*_4$. For instance, in case of $*_2$, we have $0^{*_2} = 0.25^{*_2} = 0$, $0.5^{*_2} = 0.75^{*_2} = 0.5$, and $1^{*_2} = 1$.

Example 2: There is just one hedge on the two-element Boolean algebra $\mathbf{2}$. Namely, hedge * such that $0^* = 0$ and $1^* = 1$. Hence, in case of the two-element Boolean algebra, globalization coincides with identity.

C. Fuzzy Sets and Fuzzy Relations

Suppose that $\mathbf{L}$ is our structure of truth degrees. We define usual notions: a fuzzy set (an $\mathbf{L}$-set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Operations with fuzzy sets are defined componentwise. For instance, the intersection of fuzzy sets $A, B \in \mathbf{L}^U$ is a fuzzy set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. A binary fuzzy relation (a binary $\mathbf{L}$-relation) I between X and Y is a mapping $I\colon X \times Y \rightarrow L$, $I(x, y)$ being interpreted as "the degree to which x and y are related by I". By definition, a binary fuzzy relation $I : X \times Y \rightarrow L$ is a fuzzy set in the universe $X \times Y$, i.e. $I \in \mathbf{L}^{X \times Y}$. For fuzzy sets $A, B \in \mathbf{L}^X$, put

$$S(A, B) = \textstyle\bigwedge_{x \in X}(A(x) \rightarrow B(x)), \tag{6}$$

$$A \approx B = \textstyle\bigwedge_{x \in X}(A(x) \leftrightarrow B(x)). \tag{7}$$

$S(A, B)$ and $A \approx B$ are called the degree of subsethood of A in B and the degree of equality of A and B, respectively. Note that $S(A, B)$ can be seen as a truth degree of "for each $x \in X$: if x belongs to A then x belongs to B"; similarly for $A \approx B$. $S(A, B) = 1$ iff $A(x) \leq B(x)$ for each $x \in X$ (A is fully contained in B). This fact is denoted by $A \subseteq B$.

III. Isotone Galois Connections with Hedges

A. Definition

We start by recalling the definition of and basic facts about the isotone fuzzy Galois connections [12, 17]:

Definition 1: An isotone (fuzzy) Galois connection between sets X and Y is a pair $\langle {}^\Cap, {}^\Cup \rangle$ of mappings ${}^\Cap : L^X \rightarrow L^Y$ and ${}^\Cup : L^Y \rightarrow L^X$ satisfying $S(A, B^\Cup) = S(A^\Cap, B)$.

The following theorem provides an alternative definition using perhaps more comprehensible conditions [12].

Theorem 1: A pair $\langle {}^\Cap, {}^\Cup \rangle$ of mappings ${}^\Cap : L^X \rightarrow L^Y$ and ${}^\Cup : L^Y \rightarrow L^X$ is an isotone Galois connection iff ${}^\Cap$ and ${}^\Cup$ satisfy

$$S(A_1, A_2) \leq S(A_1^\Cap, A_2^\Cap), \tag{8}$$

$$S(B_1, B_2) \leq S(B_1^\Cup, B_2^\Cup), \tag{9}$$

$$A \subseteq A^{\Cap\Cup}, \tag{10}$$

$$B \supseteq B^{\Cup\Cap}. \tag{11}$$

The importance of Galois connections, both antitone and isotone, derives from the fact that they are induced in a natural way from binary relations and that the fixpoints of Galois connections have natural meaning. A canonical way an isotone Galois connection $\langle {}^\Cap, {}^\Cup \rangle$ arises from a binary fuzzy relation I between sets X and Y is described by:

$$A^\Cap(y) = \textstyle\bigvee_{x \in X} A(x) \otimes I(x, y), \tag{12}$$

$$B^\Cup(x) = \textstyle\bigwedge_{y \in Y} I(x, y) \rightarrow B(y). \tag{13}$$

If X and Y are interpreted as the set of objects and attributes and $I(x, y)$ as a degree to which object $x \in X$ has attribute $y \in Y$, then $A^\Cap(y)$ is just the truth degree of "there exists x in A which has y" and $B^\Cup(x)$ is the truth degree of "for all y: if x has y then y belongs to B". That is, $A^\Cap$ is the fuzzy set of attributes shared by at least some object from A and $B^\Cup$ is the fuzzy set of objects whose attributes are all in B. Note that pairs of mappings (12) and (13) appear in [12, 17] and also in [16].

We now present formulas generalizing (12) and (13). Technically, we insert hedges at particular places in (12) and (13). Throughout the rest of the paper, we assume that $*_X$ and $*_Y$ are truth-stressing hedges on $\mathbf{L}$.

For a binary fuzzy relation I between X and Y we define a pair $\langle {}^\cap, {}^\cup \rangle$ of mappings ${}^\cap : L^X \rightarrow L^Y$ and ${}^\cup : L^Y \rightarrow L^X$ by

$$A^\cap(y) = \textstyle\bigvee_{x \in X} A(x)^{*_X} \otimes I(x, y), \tag{14}$$

$$B^\cup(x) = \textstyle\bigwedge_{y \in Y} I(x, y) \rightarrow B(y)^{*_Y}. \tag{15}$$

These mappings play a crucial role in our paper. The meaning of $A^\cap$ and $B^\cup$ is essentially the same as that of $A^\Cap$ and $B^\Cup$. The difference is in that parts of the verbal description of $A^\cap$ and $B^\cup$ contain "very true" compared to that of $A^\Cap$ and $B^\Cup$. For example, $A^\cap(y)$ is the truth degree of "there exists x for which it is very true that it belongs to A and which has y".

B. Basic Properties

We will use the following lemma.

Lemma 1: For a hedge *,

$$\bigvee a_i^* = (\bigvee a_i^*)^*,$$
$$(\bigwedge b_i)^* = (\bigwedge b_i^*)^*.$$

Proof: For the first equality, see [9]. For the second one, it suffices to observe that $(\bigwedge b_i)^* \leq (\bigwedge b_i^*)^*$: Since clearly $(\bigwedge b_i)^* \leq \bigwedge b_i^*$, we get $(\bigwedge b_i)^* = (\bigwedge b_i)^{**} \leq (\bigwedge b_i^*)^*$. ■

The following theorem lists the main properties of $\langle {}^\cap, {}^\cup \rangle$.

Theorem 2: Mappings $^\cap$ and $^\cup$ induced by (14) and (15) satisfy the following properties

(i) $A^\cap = A^{*_X \Cap}$ and $B^\cup = B^{*_Y \Cup}$,

(ii) $A^\cap = A^{*_X \cap}$ and $B^\cup = B^{*_Y \cup}$,

(iii) $A^\cap \subseteq A^\Cap$ and $B^\cup \subseteq B^\Cup$,

(iv) $S(A_1, A_2)^{*_X} \leq S(A_1^{*_X}, A_2^{*_X}) \leq S(A_1^\cap, A_2^\cap)$, $S(B_1, B_2)^{*_Y} \leq S(B_1^{*_Y}, B_2^{*_Y}) \leq S(B_1^\cup, B_2^\cup)$,

(v) $B^{\cup\cap} \subseteq B^{*_Y}$,

(vi) $A_1 \subseteq A_2$ implies $A_1^\cap \subseteq A_2^\cap$, $B_1 \subseteq B_2$ implies $B_1^\cup \subseteq B_2^\cup$,

(vii) $S(A^{*_X}, B^\cup) = S(A^\cap, B^{*_Y})$,

(viii) $\left(\bigcup_{i\in I} A_i^{*_X}\right)^\cap = \bigcup_{i\in I} A_i^\cap$ and $\left(\bigcap_{i\in I} B_i\right)^\cup = \left(\bigcap_{i\in I} B_i^{*_Y}\right)^\cup$,

(ix) $A^{\cap\cup\cap} \subseteq A^{\cap *_Y}$ and $B^{\cup\cap\cup} \subseteq B^{*_Y \cup}$.

Proof: (i), (ii): Follow immediately from definition of $^\cap$ and $^\cup$ and from (4).

(iii): Follows from the fact, that $\otimes$ is monotone and $\rightarrow$ is isotone in the second argument.

(iv): $S(A_1^\cap, A_2^\cap) = S(A_1^{*_X \Cap}, A_2^{*_X \Cap}) = S(A_1^{*_X}, A_2^{*_X \Cap\Cup}) \geq S(A_1^{*_X}, A_2^{*_X}) \geq S(A_1, A_2)^{*_X}$. The second assertion is similar.

(v): $B^{\cup\cap} \subseteq B^{\cup\Cap} = B^{*_Y \Cup\Cap} \subseteq B^{*_Y}$.

(vi): $A_1 \subseteq A_2$ implies $1 = S(A_1, A_2)^{*_X} \leq S(A_1^\cap, A_2^\cap)$. So $A_1^\cap \subseteq A_2^\cap$. The second claim is similar.

(vii): $S(A^{*_X}, B^\cup) = S(A^{*_X}, B^{*_Y \Cup}) = S(A^{*_X \Cap}, B^{*_Y}) = S(A^\cap, B^{*_Y})$.

(viii): Using $\bigvee a_i^{*_X} = (\bigvee a_i^{*_X})^{*_X}$, we get

$$\big(\bigcup_{i\in I} A_i^{*_X}\big)^\cap(y) = \bigvee_{x\in X} \big(\bigvee_{i\in I} A_i^{*_X}(x)\big)^{*_X} \otimes I(x,y) =$$
$$= \bigvee_{x\in X} \big(\bigvee_{i\in I} A_i^{*_X}(x)\big) \otimes I(x,y) =$$
$$= \bigvee_{i\in I}\big(\bigvee_{x\in X} A_i^{*_X}(x) \otimes I(x,y)\big) = \bigvee_{i\in I} A_i^\cap(y).$$

In a similar way, we prove the second part using $(\bigwedge b_i)^{*_Y} = (\bigwedge b_i^{*_Y})^{*_Y}$.

(ix): Using (v), we directly get $A^{\cap\cup\cap} \subseteq A^{\cap *_Y}$. Using (v) and (vi) we get $B^{\cup\cap} \subseteq B^{*_Y}$ implies $B^{\cup\cap\cup} \subseteq B^{*_Y \cup}$. ■

C. Axiomatization

We now turn to the problem of axiomatization of the mappings defined by (14) and (15). We present characteristic properties of these mappings. In what follows, we denote the mappings induced by (14) and (15) by $^{\cap_I}$ and $^{\cup_I}$. The following definition presents our characteristic conditions.

Definition 2: An isotone Galois connection with hedges ^{*_X} and ^{*_Y} between sets X and Y is a pair $\langle {}^\cap, {}^\cup \rangle$ of mappings $^\cap : L^X \rightarrow L^Y$ and $^\cup : L^Y \rightarrow L^X$ satisfying

$$S(A^{*_X}, B^\cup) = S(A^\cap, B^{*_Y}), \quad (16)$$
$$\left(\bigcup_{i\in I} A_i^{*_X}\right)^\cap = \bigcup_{i\in I} A_i^\cap, \quad (17)$$
$$\{{}^a/z\}^\cap = a^{*_X} \otimes \{{}^1/z\}^\cap. \quad (18)$$

Lemma 2: The mappings $^{\cap_I}$ and $^{\cup_I}$ defined by (14) and (15) form an isotone fuzzy Galois connection with hedges ^{*_X} and ^{*_Y}.

Proof: Due to Theorem 2 (vii) and (viii), it suffices to check $\{{}^a/z\}^\cap = a^{*_X} \otimes \{{}^1/z\}^\cap$ which directly follows from the definition of $^\cap$ using $a \otimes (\bigvee b_i) = \bigvee(a \otimes b_i)$. ■

We need the following lemma.

Lemma 3: For every mapping $^\cap : L^X \rightarrow L^Y$ there exists at most one mapping $^\cup : L^Y \rightarrow L^X$ satisfying $S(A^{*_X}, B^\cup) = S(A^\cap, B^{*_Y})$ for every $A \in L^X$ and $B \in L^Y$.

Proof: If $^{\cup'}$ is another such mapping, we have $S(A^{*_X}, B^\cup) = S(A^{*_X}, B^{\cup'})$ for any A and B. Take any $x \in X$ and put $A = \{{}^1/x\}$. Then

$$B^\cup(x) = S(A^{*_X}, B^\cup) = S(A^{*_X}, B^{\cup'}) = B^{\cup'}(x).$$

Therefore, $^\cup$ coincides with $^{\cup'}$. ■

Lemma 4: Let $\langle {}^\cap, {}^\cup \rangle$ be an isotone Galois connection with hedges ^{*_X} and ^{*_Y}. Then there exists an **L**-relation I between X and Y such that $\langle {}^\cap, {}^\cup \rangle = \langle {}^{\cap_I}, {}^{\cup_I} \rangle$.

Proof: We need to find I such that $A^\cap = A^{\cap_I}$ and $B^\cup = B^{\cup_I}$ for all $A \in L^X$, $B \in L^Y$. Due to Lemma 3, it is sufficient to find I for which $A^\cap = A^{\cap_I}$. Namely, $\langle {}^{\cap_I}, {}^{\cup_I} \rangle$ satisfy $S(A^{*_X}, B^{\cup_I}) = S(A^{\cap_I}, B^{*_Y})$ by Lemma 2. Hence, $^{\cup_I}$ coincides with $^\cup$ due to Lemma 3.

Define I by $I(x,y) = \{{}^1/x\}^\cap(y)$. Then using (17) and (18) we get

$$A^\cap(y) = A^{*_X \cap}(y) = \big(\bigcup_{x\in X} \{{}^{A^{*_X}(x)}/x\}\big)^\cap(y) =$$
$$= \big(\bigcup_{x\in X} \{{}^{A(x)}/x\}^\cap\big)(y) = \bigvee_{x\in X} \{{}^{A(x)}/x\}^\cap(y) =$$
$$= \bigvee_{x\in X} A(x)^{*_X} \otimes \{{}^1/x\}^\cap(y) =$$
$$= \bigvee_{x\in X} A(x)^{*_X} \otimes I(x,y) = A^{\cap_I}(y).$$

This finishes the proof. ■

The following theorem presents the main result of this section: Every fuzzy relation induces an isotone Galois connection with hedges, every isotone Galois connection with hedges is induced by a fuzzy relation, and such relationship is one-to-one.

Theorem 3: Let I be an **L**-relation between X and Y, $\langle {}^\cap, {}^\cup \rangle$ be an isotone Galois connection with hedges $*_X$ and $*_Y$. Then

(i) $\langle {}^{\cap_I}, {}^{\cup_I} \rangle$ is an isotone Galois connection with hedges $*_X$ and $*_Y$;

(ii) $I_{\langle \cap, \cup \rangle}$ defined by $I_{\langle \cap, \cup \rangle}(x,y) = \{{}^1/x\}^\cap(y)$ is an **L**-relation between X and Y and we have

(iii) $\langle {}^{\cap}, {}^{\cup} \rangle = \left\langle {}^{\cap_{I_{\langle \cap, \cup \rangle}}}, {}^{\cup_{I_{\langle \cap, \cup \rangle}}} \right\rangle$ and $I = I_{\langle {}^{\cap_I}, {}^{\cup_I} \rangle}$.

Proof: Due to Lemma 2 and Lemma 4, it suffices to prove $I = I_{\langle {}^{\cap_I}, {}^{\cup_I} \rangle}$. We have

$$I_{\langle {}^{\cap_I}, {}^{\cup_I} \rangle}(x, y) = \{{}^{1}/x\}^{\cap_I}(y) =$$
$$= \bigvee_{z \in X} \{{}^{1^{*_X}}/x\}(z) \otimes I(z, y) = I(x, y).$$

■

IV. Associated Concept Lattices and Further Issues

Sets of fixed points of Galois connections are called concept lattices in formal concept analysis [11]. Given $\langle X, Y, I \rangle$ (input data, objects, attributes and relationship between them), we are interested in the set of all fixpoints of the isotone Galois connections associated to $\langle X, Y, I \rangle$, i.e. in the set

$$\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I) = \{\langle A, B \rangle \mid A^{\cap_I} = B, B^{\cup_I} = A\}.$$

$\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$ is called the concept lattice with hedges associated to $\langle X, Y, I \rangle$. Elements $\langle A, B \rangle$ of $\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$ are called formal concepts. If both ^{*_X} and ^{*_Y} are identities, this concept lattice coincides with the one from [17]. The structure of our concept lattices with hedges will be presented in a follow-up paper. Likewise, we will present theoretical results on how the choice of hedges influences the number of formal concepts in $\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$. In this paper, we confine ourselves to illustrative examples demonstrating the role of hedges.

V. Example

We now illustrate the influence of hedges by the following example. Consider a fuzzy relation represented by Table I. The table describes six books and their graded attributes like “low price”, “high book rating”. If we take the five-value Łukasiewicz chain $\mathbf{L} = \langle \{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \rightarrow, 0, 1 \rangle$ as our structure of truth degrees, we can consider $5 \times 5 = 25$ combinations of hedges $*_X$ and $*_Y$ on $\mathbf{L}$. Recall that the hedges for $\mathbf{L}$ are depicted in Fig. 1. For each combination of hedges $*_X$ and $*_Y$, we obtain one concept lattice $\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$ of fixed points of the corresponding ${}^{\cap}$ and ${}^{\cup}$ defined by (14) and (15), respectively. The lattices of fixed points are depicted in Fig. 2; rows in the figure correspond to $*_X$ while columns correspond to $*_Y$. As we can see from the picture, the reduction of the number of fixed points (concepts) is quite extensive.

Observe an interesting fact in the example. Namely, $*_Y$ has a greater impact on the number of fixed points in $\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$. This behavior can be explain by looking at the strongest $*_Y$, i.e. at $*_Y$ being globalization, as follows. If $*_Y$ is globalization and $\langle X, Y, I \rangle$ contains at least one 1 in each row, i.e., if for each $x \in X$ there is $y \in Y$ such that $I(x, y) = 1$, then $B^{\cup} = \emptyset$ whenever $\{y \in Y \mid B(y) = 1\} = \emptyset$. Indeed, in such a case, we have $B^{\cup}(x) = \bigwedge_{y \in Y} I(x, y) \rightarrow B(y)^{*_Y} = \bigwedge_{y \in Y} I(x, y) \rightarrow 0 \leq 1 \rightarrow 0 = 0$. Therefore, B cannot be an intent unless $B = \emptyset$. A further study of the influence of hedges on the size of $\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$ is the subject of further research.

TABLE I

Context of books and their graded properties

	High Rating	Large No. of Pages	Low Price	Top Sales Rank
1	0.75	0.00	1.00	0.00
2	0.50	1.00	0.25	0.50
3	1.00	1.00	0.25	0.50
4	0.75	0.50	0.25	1.00
5	0.75	0.25	0.75	0.00
6	1.00	0.00	0.75	0.25

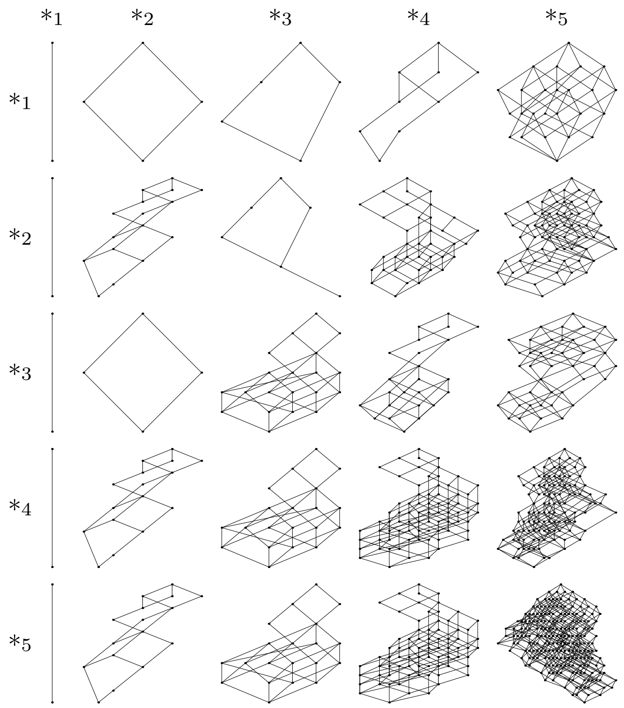

Fig. 2. Lattices of fixed points of isotone galois connections with hedges induced by Table I. The picture shows lattices of fixed points resulting by all combinations of hedges $*_X$ and $*_Y$ from Fig. 1.

VI. Conclusions and Future Research

We developed foundations of isotone Galois connections with hedges. We focused on the basic calculus of such connections, i.e. on the properties analogous to those which are essential for the other type of Galois connections studied in the literature. Further research will include the following topics:

– Study of $\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$. Particularly, an analogy of basic theorem of concept lattices for our setting.

– Results on how hedges influence the number of formal concepts in $\mathcal{B}(X^{*_X \cap}, Y^{*_Y \cup}, I)$.

– Study of related structures such as closure structures, attribute dependencies, etc. which naturally correspond to isotone Galois connections with hedges. A paper on this topic is in preparation.

– Optimal decompositions of matrices. We proved in our earlier work that fixpoints of Galois connections can be used to find optimal decompositions of matrices with degrees, in that the inner dimension in the decomposition is minimal [7]. Fixpoints of isotone Galois connections serve for triangular decompositions of matrices with degrees. Inclusion of hedges introduces additional constraints. For example, we can require that one of the matrices into which we decompose a given matrix is binary one. Investigation of this topic is in progress.

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