AISTA 2004 in Cooperation with the IEEE Computer Society Proceedings / ISBN : 2-9599776-8-8
15-18 November, Kirchberg - Luxembourg

Implications from data with fuzzy attributes

Radim Bělohlávek, Martina Chlupová, and Vilém Vychodil
Dept. Computer Science, Palacký University, Tomkova 40, CZ-779 00, Olomouc, Czech Republic
Email: {radim.belohlavek, martina.chlupova, vilem.vychodil}@upol.cz

***Abstract*— We present an algorithm for generating a complete and non-redundant set of attribute implications from data table with fuzzy attributes. Rows and columns in the data table correspond to objects and attributes, respectively. An attribute implication is an expression which says that if an object satisfies a certain collection of attributes then it satisfies some other collection of attributes as well. The algorithm is based on reduction of the problem to the problem of generating fixed points of a fuzzy closure operator. We describe the theoretical insight, the algorithm, and examples.**

I. Introduction and problem setting

Tabular data describing objects and their attributes represents a basic form of data. Several methods have been designed to analyze object-attribute data. Among these, methods for obtaining if-then rules (implications) from data are of the most popular ones. For example, the well-known mining of association rules is an example of rule-extraction method, see see [1] and also [13]. The notion of a rule and its validity in tabular data has been defined in several ways.

In our paper, we are interested in if-then rules generated from data with fuzzy attributes: rows and columns of data table correspond to objects $x \in X$ and attributes $y \in Y$, respectively. Table entries $I(x, y)$ are truth degrees to which object x has attribute y. Unlike the classical case, we assume that $I(x, y)$ is taken from a suitable scale L of truth degrees, e.g. $L = [0, 1]$. We are interested in rules of the form "if A then B" ($A \Rightarrow B$), where A and B are collections of attributes, with the meaning: if an object has all the attributes of A then it has also all attributes of B. In crisp case, these rules were thoroughly investigated, see e.g. [10] for the first paper and [9] for further information and references. Our aim is basically to look at such if-then rules from the point of view of fuzzy logic. Our motivation is threefold: (1) in practice, attributes are usually fuzzy rather than bivalent; (2) non-logical attributes (like age, etc.) can be scaled to fuzzy attributes; (3) to investigate connections with related methods for processing of data with fuzzy attributes, particularly with formal concept analysis, e.g. [3], [8].

We study the following topics: appropriate tractable definition of if-then rules $A \Rightarrow B$ and their semantics (validity degree etc.); directly related mathematical structures; the notion of semantic entailment of if-then rules with the aim to obtain a non-redundant basis of all valid rules; algorithms for generating bases.

II. Preliminaries

As a set of truth degrees equipped with suitable operations (truth functions of logical connectives) we use a so-called complete residuated lattice with truth-stressing hedge. A complete residuated lattice with truth-stressing hedge (shortly, a hedge) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \le c \quad \text{iff} \quad a \le b \rightarrow c \tag{1}$$

for each $a, b, c \in L$; hedge * satisfies

$$a^* \le a, \tag{2}$$
$$(a \rightarrow b)^* \le a^* \rightarrow b^*, \tag{3}$$
$$a^{**} = a^*, \tag{4}$$
$$\textstyle\bigwedge_{i\in I} a_i^* = \left(\bigwedge_{i \in I} a_i\right)^* \tag{5}$$

for each $a, b \in L$, $a_i \in L$ ($i \in I$). Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [11], [12]. Properties (2)–(5) have natural interpretations, e.g. (2) can be read: "if a is very true, then a is true", (3) can be read: "if $a \rightarrow b$ is very true and if a is very true, then b is very true", etc.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

Łukasiewicz:
$$\begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \rightarrow b &= \min(1 - a + b, 1), \end{aligned} \tag{6}$$

Gödel:
$$\begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \le b, \\ b & \text{otherwise,} \end{cases} \end{aligned} \tag{7}$$

Goguen (product):
$$\begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \le b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{aligned} \tag{8}$$

In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0, 1]$ to L. A special case of both of these chains is the Boolean algebra with $L = \{0, 1\}$ (structure of truth degrees of classical logic).

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization [14]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Having $\mathbf{L}$ as our structure of truth degrees, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1, \dots, u_n\}$ then A can be denoted by $A = \{{}^{a_1}/u_1, \dots, {}^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1, \dots, n$. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u \in U} \big(A(u) \rightarrow B(u)\big), \tag{10}$$

which generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary $\mathbf{L}$-relation on $\mathbf{L}^U$). Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [3], [11].

III. Fuzzy attribute implications

A. Definition, validity, and basic properties

Fuzzy attribute implication (*over attributes* Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). The intended meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has also all attributes from B". The notions "being very true", "to have an attribute", and logical connective "if-then" are determined by the chosen $\mathbf{L}$.

For an $\mathbf{L}$-set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M:

$$||A \Rightarrow B||_M = S(A, M)^* \rightarrow S(B, M). \tag{11}$$

If M is the fuzzy set of all attributes of an object x, then $||A \Rightarrow B||_M$ is the truth degree to which $A \Rightarrow B$ holds for x.

Fuzzy attribute implications can be used to describe dependencies in data tables with fuzzy attributes. Let X and Y be sets of objects and attributes, respectively, I be an $\mathbf{L}$-relation between X and Y, i.e. I is a mapping $I : X \times Y \rightarrow L$. $\langle X, Y, I\rangle$ is called a *data table with fuzzy attributes.* $\langle X, Y, I\rangle$ represents a table which assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ to which object x has attribute y. If both X and Y are finite, $\langle X, Y, I\rangle$ can be visualized as in Fig. 1

For fuzzy sets $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$ (i.e. A is a fuzzy set of objects, B is a fuzzy set of attributes), we define fuzzy sets

I	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$	
x	$\cdots$	$I(x, y)$	$\cdots$
$\vdots$		$\vdots$	

Fig. 1. Data table with fuzzy attributes

$A^{\uparrow} \in \mathbf{L}^Y$ (fuzzy set of attributes), $B^{\downarrow} \in \mathbf{L}^X$ (fuzzy set of objects) by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x \in X} \big(A(x)^* \rightarrow I(x, y)\big), \tag{12}$$

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y \in Y} \big(B(y) \rightarrow I(x, y)\big). \tag{13}$$

We put

$$\mathcal{B}(X^*, Y, I) = \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

and define for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^*, Y, I)$ a binary relation $\leq$ by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). Operators ${}^{\downarrow}, {}^{\uparrow}$ form so-called Galois connection with hedge, see [6]. The structure $\langle \mathcal{B}(X^*, Y, I), \leq\rangle$ is called a *fuzzy concept lattice* induced by $\langle X, Y, I\rangle$. The elements $\langle A, B\rangle$ of $\mathcal{B}(X^*, Y, I)$ are naturally interpreted as concepts (clusters) hidden in the input data represented by I. Namely, $A^{\uparrow} = B$ and $B^{\downarrow} = A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. Note that these conditions represent exactly the definition of a concept as developed in the so-called Port-Royal logic; A and B are called the *extent* and the *intent* of the concept $\langle A, B\rangle$, respectively, and represent the collection of all objects and all attributes covered by the particular concept. Furthermore, $\leq$ models the natural subconcept-superconcept hierarchy—concept $\langle A_1, B_1\rangle$ is a subconcept of $\langle A_2, B_2\rangle$ iff each object from A_1 belongs to A_2 (dually for attributes).

Now we define a validity degree of fuzzy attribute implications in data tables and intents of fuzzy concept lattices. First, for a set $\mathcal{M} \subseteq \mathbf{L}^Y$ (i.e. $\mathcal{M}$ is an ordinary set of $\mathbf{L}$-sets) we define a degree $||A \Rightarrow B||_{\mathcal{M}} \in L$ to which $A \Rightarrow B$ holds in $\mathcal{M}$ by

$$||A \Rightarrow B||_{\mathcal{M}} = \textstyle\bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M. \tag{14}$$

Having $\langle X, Y, I\rangle$, let $I_x \in \mathbf{L}^Y$ ($x \in X$) be $\mathbf{L}$-set of attributes such that $I_x(y) = I(x, y)$ for each $y \in Y$. Described verbally, I_x is the $\mathbf{L}$-set of all attributes of $x \in X$, i.e. in $\langle X, Y, I\rangle$, I_x corresponds to a row labeled x. Clearly, we have $I_x = \{{}^1/x\}^{\uparrow}$ for each $x \in X$.

A *degree* $||A \Rightarrow B||_{\langle X,Y,I\rangle} \in L$ *to which* $A \Rightarrow B$ *holds in* (*each row of*) $\langle X, Y, I\rangle$ is defined by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{M}}, \tag{15}$$

where $\mathcal{M} = \{I_x \mid x \in X\}$.

Denote

$$\mathrm{Int}(X^*, Y, I) = \{B \in \mathbf{L}^Y \mid \langle A, B\rangle \in \mathcal{B}(X^*, Y, I) \text{ for some } A\}$$

the set of all intents of concepts of $\mathcal{B}(X^*, Y, I)$. Since $M \in \mathbf{L}^Y$ is an intent of some concept of $\mathcal{B}(X^*, Y, I)$ iff $M = M^{\downarrow\uparrow}$, we have $\mathrm{Int}(X^*, Y, I) = \{M \in \mathbf{L}^Y \mid M = M^{\downarrow\uparrow}\}$.

A *degree* $||A \Rightarrow B||_{\mathcal{B}(X^*,Y,I)} \in L$ *to which* $A \Rightarrow B$ *holds in (intents of)* $\mathcal{B}(X^*, Y, I)$ is defined by

$$||A \Rightarrow B||_{\mathcal{B}(X^*,Y,I)} = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}. \qquad (16)$$

The following lemma connecting the degree to which $A \Rightarrow B$ holds in $\langle X, Y, I\rangle$, the degree to which $A \Rightarrow B$ holds in intents of $\mathcal{B}(X^*, Y, I)$, and the degree of subsethood of B in $A^{\downarrow\uparrow}$ was proved in [7].

Lemma. *Let* $\langle X, Y, I\rangle$ *be a data table with fuzzy attributes. Then*

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathcal{B}(X^*,Y,I)} = S(B, A^{\downarrow\uparrow}) \qquad (17)$$

for each fuzzy attribute implication $A \Rightarrow B$. ■

B. Implication bases

The main obstacle to extract fuzzy concepts from data tables with fuzzy attributes is that large data tables and fine scales of truth degrees usually lead to large amounts of concepts which are then not graspable by our mind (it is unlikely to benefit from thousands of concepts, because we would have serious problems just trying to read them). It is then interesting to describe concepts as models of (possibly smaller) sets of fuzzy attribute implications.

Let $\langle X, Y, I\rangle$ be a data table with fuzzy attributes and let T be a set of fuzzy attribute implications. $M \in \mathbf{L}^Y$ is called a *model* of T if $||A \Rightarrow B||_M = 1$ for each $A \Rightarrow B \in T$. The set of all models of T is denoted by $\mathrm{Mod}(T)$, i.e.

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid M \text{ is a model of } T\}. \qquad (18)$$

A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ *semantically follows* from T is defined by

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Mod}(T)}. \qquad (19)$$

T is called *complete* (in $\langle X, Y, I\rangle$) if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$ for each $A \Rightarrow B$. If T is complete and no proper subset of T is complete, then T is called a *non-redundant basis*. Note that the notion of completeness of T depends on a given data table with fuzzy attributes.

The following assertion shows that the models of a complete set of fuzzy attribute implications are exactly the intents of the corresponding fuzzy concept lattice.

Theorem. T *is complete iff* $\mathrm{Mod}(T) = \mathrm{Int}(X^*, Y, I)$.

Proof: Let T be complete. Suppose $M \in \mathrm{Mod}(T)$. We have $||M \Rightarrow M^{\downarrow\uparrow}||_{\mathrm{Int}(X^*,Y,I)} = S(M^{\downarrow\uparrow}, M^{\downarrow\uparrow}) = 1$ by (17), i.e. $||M \Rightarrow M^{\downarrow\uparrow}||_T = 1$ by completeness and (17). Since M is a model of T, we have $||M \Rightarrow M^{\downarrow\uparrow}||_M = 1$ which immediately gives $1 = S(M, M)^* \leq S(M^{\downarrow\uparrow}, M)$, i.e. $M^{\downarrow\uparrow} \subseteq M$. That is, $M \in \mathrm{Int}(X^*, Y, I)$ which proves that $\mathrm{Mod}(T) \subseteq \mathrm{Int}(X^*, Y, I)$. Now take $M \in \mathrm{Int}(X^*, Y, I)$. For each implication $A \Rightarrow B \in T$ we have $||A \Rightarrow B||_M \geq ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = ||A \Rightarrow B||_{\mathrm{Mod}(T)} = 1$ by (17), i.e. $M \in \mathrm{Mod}(T)$ showing $\mathrm{Int}(X^*, Y, I) \subseteq \mathrm{Mod}(T)$.

Conversely, if $\mathrm{Mod}(T) = \mathrm{Int}(X^*, Y, I)$ then $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$ by (17). ■

Therefore, we are interested in finding non-redundant bases. First, a non-redundant basis T is a minimal set of implications which conveys, via the notion of semantic entailment, information about validity of attribute implications in $\langle X, Y, I\rangle$. In particular, attribute implications which are true (in degree 1) in $\langle X, Y, I\rangle$ are exactly those which follow (in degree 1) from T. Second, non-redundant bases are promising candidates for being the minimal complete sets of attribute implications which describe the concept intents (and consequently, the whole fuzzy concept lattice).

IV. Algorithm for getting non-redundant bases

A. Systems of pseudo-intents

Given $\langle X, Y, I\rangle$, $\mathcal{P} \subseteq \mathbf{L}^Y$ (a system of fuzzy sets of attributes) is called a *system of pseudo-intents* of $\langle X, Y, I\rangle$ if for each $P \in \mathbf{L}^Y$ we have:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P.$$

If * is globalization and if Y is finite, then for each $\langle X, Y, I\rangle$ there exists a unique system of pseudo-intents (this is not so for the other hedges in general). From now on, let Y be finite and let $\mathbf{L}$ be finite and linearly ordered. Moreover, for $Z \in \mathbf{L}^Y$ we put

$$Z^{T^*} = Z \cup \bigcup\{B \otimes S(A, Z)^* \mid A \Rightarrow B \in T \text{ and } A \neq Z\},$$
$$Z^{T^*_0} = Z,$$
$$Z^{T^*_n} = (Z^{T^*_{n-1}})^{T^*}, \quad \text{for } n \geq 1,$$

and define an operator cl_{T^*} on $\mathbf{L}$-sets in Y by

$$cl_{T^*}(Z) = \textstyle\bigcup_{n=0}^{\infty} Z^{T^*_n}. \qquad (20)$$

Theorem. *Let* $\mathcal{P}$ *be a system of pseudo-intents and put*

$$T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}. \qquad (21)$$

(i) T *is non-redundant basis. Moreover, if* * *is globalization then* T *is minimal.* (ii) *If* * *is globalization then* cl_{T^*} *is an* $\mathbf{L}^*$*-closure operator, and*

$$\{cl_{T^*}(Z) \mid Z \in \mathbf{L}^Y\} = \mathcal{P} \cup \mathrm{Int}(X^*, Y, I). \qquad (22)$$

Proof: For (i) see [7].

(ii): We check that cl_{T^*} satisfies the following properties of $\mathbf{L}^*$-closure operators

$$Z \subseteq cl_{T^*}(Z), \qquad (23)$$
$$S(Z_1, Z_2)^* \leq S(cl_{T^*}(Z_1), cl_{T^*}(Z_2)), \qquad (24)$$
$$cl_{T^*}(Z) = cl_{T^*}(cl_{T^*}(Z)), \qquad (25)$$

for each $Z, Z_1, Z_2 \in \mathbf{L}^Y$. (23) is obvious.

(24): Since * is globalization and T is defined by (21), we have $Z^{T^*} = Z \cup \bigcup\{Q^{\downarrow\uparrow} \mid Q \in \mathcal{P},\, Q \subset Z\}$. If $S(Z_1, Z_2)^* = 1$ then $Z_1 \subseteq Z_2$, i.e. if $Q \subset Z_1$, then $Q \subset Z_2$ yielding ${Z_1}^{T^*} \subseteq {Z_2}^{T^*}$, i.e. $S(cl_{T^*}(Z_1), cl_{T^*}(Z_2)) = 1$.

In order to show (25), it is sufficient to check $(cl_{T^*}(Z))^{T^*} \subseteq cl_{T^*}(Z)$. If $Q \subset cl_{T^*}(Z)$ then $Q \subset Z^{T^*_n}$ for some $n \in \mathbb{N}_0$ (recall that $\mathbf{L}$ is finite and linearly ordered, and Y is finite). Hence, $Q^{\downarrow\uparrow} \subseteq Z^{T^*_{n+1}} \subseteq cl_{T^*}(Z)$. That is, $(cl_{T^*}(Z))^{T^*} \subseteq cl_{T^*}(Z)$ yielding $cl_{T^*}(cl_{T^*}(Z)) \subseteq cl_{T^*}(Z)$. Altogether, cl_{T^*} is an $\mathbf{L}^*$-closure operator.

$\mathcal{P} \cup \mathrm{Int}(X^*, Y, I) \subseteq \{cl_{T^*}(Z) \mid Z \in \mathbf{L}^Y\}$ follows directly from properties of globalization. For the converse inclusion it suffices to show that if $cl_{T^*}(Z) \neq cl_{T^*}(Z)^{\downarrow\uparrow}$ then $cl_{T^*}(Z)$ is in $\mathcal{P}$. Consider $Q \in \mathcal{P}$ with $Q \subset cl_{T^*}(Z)$. Then $Q^{\downarrow\uparrow} \subseteq (cl_{T^*}(Z))^{T^*} = cl_{T^*}(Z)$, i.e. $||Q \Rightarrow Q^{\downarrow\uparrow}||_{cl_{T^*}(Z)} = 1$. ■

For general hedge *, operator cl_{T^*} need not satisfy the monotony condition (24). Consider a finite Łukasiewicz chain $\mathbf{L}$ with $L = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ and let * be the identity ($a^* = a$). Take $T = \{\{^{0.1}/y\} \Rightarrow \{^{1}/y\}\}$. Obviously, $cl_{T^*}(\{^{0}/y\})(y) \geq \left(\{^{0}/y\}^{T^*}\right)(y) = 0 \vee (1 \otimes (0.1 \rightarrow 0)) = 0.9$. On the other hand, $cl_{T^*}(\{^{0.1}/y\})(y) = 0.1$. Thus, we have $\{^{0}/y\} \subseteq \{^{0.1}/y\}$, and $cl_{T^*}(\{^{0}/y\}) \not\subseteq cl_{T^*}(\{^{0.1}/y\})$, i.e. cl_{T^*} is not monotone.

B. Algorithm for getting all (pseudo) intents

The previous theorem showed that for * being the globalization, we can get all intents and all pseudo-intents (of a given data table with fuzzy attributes) by computing the fixed points of cl_{T^*}. This can be done with polynomial time delay using the fuzzy extension of Ganter's algorithm for computing all fixed points of a closure operator, see [5].

Hence, if $\mathbf{L}$ is finite and linearly ordered residuated lattice with * being the globalization, one can compute all (pseudo) intents of $\langle X, Y, I\rangle$ (Y being finite) using the following algorithm:

Input: $\langle X, Y, I\rangle$.
Output: $\mathrm{Int}(X^*, Y, I)$ (intents), $\mathcal{P}$ (pseudo-intents).

$B := \emptyset$
if $B = B^{\downarrow\uparrow}$:
  add B *to* $\mathrm{Int}(X^*, Y, I)$
else:
  add B *to* $\mathcal{P}$
while $B \neq Y$:
  $T := \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$
  $B := B^+$
  if $B = B^{\downarrow\uparrow}$:
    add B *to* $\mathrm{Int}(X^*, Y, I)$
  else:
    add B *to* $\mathcal{P}$

Note that B^+ denotes the lectically smallest fixed point of $\mathbf{L}$-closure operator cl_{T^*} which is a successor of B. For more details on extended Ganter's algorithm, we refer to [5].

V. Illustrative example and remarks

Let $\mathbf{L}$ be a three-element Łukasiewicz chain such that $\mathbf{L}$ consists of $L = \{0, 0.5, 1\}$ ($0 < 0.5 < 1$) endowed with $\otimes, \rightarrow$ defined by (6), and let * be the globalization. The data table is given by Table I. The set X of object consists of objects

TABLE I
Data table with fuzzy attributes

	size		distance	
	small (s)	large (l)	far (f)	near (n)
Mercury (Me)	1	0	0	1
Venus (Ve)	1	0	0	1
Earth (Ea)	1	0	0	1
Mars (Ma)	1	0	0.5	1
Jupiter (Ju)	0	1	1	0.5
Saturn (Sa)	0	1	1	0.5
Uranus (Ur)	0.5	0.5	1	0
Neptun (Ne)	0.5	0.5	1	0
Pluto (Pl)	1	0	1	0

TABLE II
Extracted concepts

	extent									intent			
no.	Me	Ve	Ea	Ma	Ju	Sa	Ur	Ne	Pl	s	l	f	n
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	0.5	0	0	0.5	0.5	1	1	0	1	0
3.	0.5	0.5	0.5	1	0	0	0	0	0	1	0	0.5	1
4.	0.5	0.5	0.5	1	0	0	0.5	0.5	1	1	0	0.5	0
5.	1	1	1	1	0	0	0	0	0	1	0	0	1
6.	1	1	1	1	0	0	0.5	0.5	1	1	0	0	0
7.	0	0	0	0.5	0.5	0.5	1	1	0.5	0.5	0.5	1	0
8.	0	0	0	0.5	0.5	0.5	1	1	1	0.5	0	1	0
9.	0.5	0.5	0.5	1	0.5	0.5	1	1	1	0.5	0	0.5	0
10.	1	1	1	1	0.5	0.5	1	1	1	0.5	0	0	0
11.	0	0	0	0	1	1	0.5	0.5	0	0	1	1	0.5
12.	0	0	0	0.5	1	1	1	1	0.5	0	0.5	1	0
13.	0	0	0	0.5	1	1	1	1	1	0	0	1	0
14.	0.5	0.5	0.5	1	1	1	0.5	0.5	0.5	0	0	0.5	0.5
15.	0.5	0.5	0.5	1	1	1	1	1	1	0	0	0.5	0
16.	1	1	1	1	1	1	0.5	0.5	0.5	0	0	0	0.5
17.	1	1	1	1	1	1	1	1	1	0	0	0	0

"Mercury", "Venus", …, Y contains four attributes: size of the planet (small / large), distance from the sun (far / near). The corresponding fuzzy concepts (clusters) extracted from this data table are identified in Table II, where each row represents a single concept. The subconcept-superconcept hierarchy (fuzzy concept lattice) is depicted in Fig. 2.

Concepts in Table II have natural interpretation. For instance concept 2. can be understood as cluster of "small planets far from sun", concept 14. can be interpreted as cluster of "planets with average distance from sun". Concepts 1. and 17. represent borderline concepts.

Before we present the non-redundant basis, let us introduce the following convention for writing finite $\mathbf{L}$-fuzzy sets: we write $\{\ldots, u, \ldots\}$ instead of $\{\ldots, {}^{1}/u, \ldots\}$, and we also omit elements of U whose membership degree is zero. For example, we write $\{u, {}^{0.5}/v\}$ instead of $\{{}^{1}/u, {}^{0.5}/v, {}^{0}/w\}$, etc.

The system $\mathcal{P}$ of pseudo-intents is the following

$$\mathcal{P} = \{\{s, {}^{0.5}/l, f\}, \{{}^{0.5}/s, {}^{0.5}/n\}, \{l, f\}, \{{}^{0.5}/l\}, \{f, {}^{0.5}/n\}, \{n\}\}.$$

Hence, the minimal non-redundant basis T defined by (21)

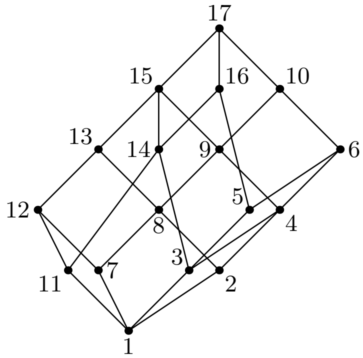

Fig. 2. Fuzzy concept lattice

TABLE III

AVERAGE SIZES OF BASES (THREE TRUTH DEGREES)

	3 attributes			5 attributes			7 attributes		
objs.	$\mathcal{B}$	$\mathcal{P}$	ratio	$\mathcal{B}$	$\mathcal{P}$	ratio	$\mathcal{B}$	$\mathcal{P}$	ratio
5	9	5	0.556	13	11	0.846	17	17	1.000
10	15	4	0.267	34	14	0.412	58	26	0.448
15	19	4	0.211	56	17	0.304	123	39	0.317
20	21	3	0.143	75	18	0.240	164	48	0.293
25	23	3	0.130	89	18	0.202	238	56	0.235
30	24	3	0.125	105	19	0.181	303	66	0.218
35	25	2	0.080	119	19	0.160	385	67	0.174
40	26	2	0.077	125	19	0.152	391	70	0.179

consists of the following fuzzy attribute implications:

$$\{s, {}^{0.5}\!/l, f\} \Rightarrow \{s, l, f, n\}, \qquad \{{}^{0.5}\!/s, {}^{0.5}\!/n\} \Rightarrow \{s, n\},$$
$$\{l, f\} \Rightarrow \{l, f, {}^{0.5}\!/n\}, \qquad \{{}^{0.5}\!/l\} \Rightarrow \{{}^{0.5}\!/l, f\},$$
$$\{f, {}^{0.5}\!/n\} \Rightarrow \{l, f, {}^{0.5}\!/n\}, \qquad \{n\} \Rightarrow \{s, n\}.$$

Minimal non-redundant bases are not given uniquely in general. For example, the previous set T can be converted into so-called *stem basis* $T^{\circ} = \{P \Rightarrow P^{\circ} \mid P \in \mathcal{P}\}$, where

$$P^{\circ}(y) = \begin{cases} 0 & \text{if } P^{\downarrow\uparrow}(y) = P(y), \\ P^{\downarrow\uparrow}(y) & \text{otherwise,} \end{cases}$$

for each $P \in \mathcal{P}$, $y \in Y$. It is easily seen that T°:

$$\{s, {}^{0.5}\!/l, f\} \Rightarrow \{l, n\}, \qquad \{{}^{0.5}\!/s, {}^{0.5}\!/n\} \Rightarrow \{s, n\},$$
$$\{l, f\} \Rightarrow \{{}^{0.5}\!/n\}, \qquad \{{}^{0.5}\!/l\} \Rightarrow \{f\},$$
$$\{f, {}^{0.5}\!/n\} \Rightarrow \{l\}, \qquad \{n\} \Rightarrow \{s\}$$

is also a minimal non-redundant basis. Note that the implications in both bases represent information which completely describes all concepts. For instance, $\{n\} \Rightarrow \{s\}$ can be read: "each near planet is small", $\{{}^{0.5}\!/l\} \Rightarrow \{f\}$ can be read: "if x is large in degree 0.5, then x is far". A finer reading of fuzzy attribute implications depends on the interpretation of the truth degrees, on the chosen structure of truth degrees, and of course, on the particular application.

To sum up, six fuzzy attribute implications are sufficient to determine 17 concepts of a data table with fuzzy attributes of size 9×4 with three truth degrees.

TABLE IV

AVERAGE SIZES OF BASES (FIVE TRUTH DEGREES)

	3 attributes			5 attributes			7 attributes		
objs.	$\mathcal{B}$	$\mathcal{P}$	ratio	$\mathcal{B}$	$\mathcal{P}$	ratio	$\mathcal{B}$	$\mathcal{P}$	ratio
5	12	9	0.750	17	19	1.118	25	26	1.040
10	26	12	0.462	69	29	0.420	130	51	0.392
15	40	12	0.300	139	38	0.273	358	77	0.215
20	52	12	0.231	216	47	0.218	591	102	0.173
25	66	12	0.182	322	54	0.168	853	129	0.151
30	71	12	0.169	421	59	0.14	1326	166	0.125
35	76	11	0.145	505	64	0.127	2115	207	0.098
40	81	11	0.136	575	68	0.118	2356	234	0.099

Experimental results have shown that the number of implications is usually (considerably) smaller than the number of concepts. However, the number of implications varies depending on density of the input data table (sparse tables can lead to relatively small amounts of concepts but large amounts of implications). Tables III and IV contain a summary of average number of (pseudo) intents of randomly generated data tables with 3, 5, or 7 attributes and with 5 up to 40 objects (columns labeled "$\mathcal{B}$" contain average number of concepts, columns labeled "$\mathcal{P}$" contain average size of the minimal bases, columns labeled "ratio" contain quotient of the previous values).

ACKNOWLEDGMENT

Research of R. Bělohlávek and V. Vychodil supported by grant no. B1137301 of GA AV ČR. Research of R. Bělohlávek supported by grant No. 201/02/P076 of the Czech Science Foundation.

REFERENCES

[1] Adamo J.-M.: *Data Mining for Association Rules and Sequential Patterns. Sequential and Parallel Alorithms.* Springer, New York, 2001.
[2] Bělohlávek R.: Fuzzy Galois connections. *Math. Logic Quarterly* **45**,4 (1999), 497–504.
[3] Bělohlávek R.: *Fuzzy Relational Systems: Foundations and Principles*. Kluwer, Academic/Plenum Publishers, New York, 2002.
[4] Bělohlávek R.: Concept lattices and order in fuzzy logic. *Ann. Pure Appl. Logic* **128**(2004), 277–298.
[5] Bělohlávek R.: Getting maximal rectangular submatrices from $[0, 1]$-valued object-attribute tables: algorithms for fuzzy concept analysis (submitted). Preliminary version in *Proc. Fourth Int. Conf. on Recent Advances in Soft Computing*. Nottingham, United Kingdom, 12-13 December, 2002, pp. 200–205.
[6] Bělohlávek R., Funioková T., Vychodil V.: Galois connections with hedges. IFSA Congress 2005 (submitted).
[7] Bělohlávek R., Vychodil V.: Fuzzy attribute logic: attribute implications, their validity, entailment, and non-redundant basis. IFSA Congress 2005 (submitted).
[8] Burusco A., Fuentes-Gonzales R.: The study of L-fuzzy concept lattice. *Mathware & Soft Computing* **3**(1994), 209–218.
[9] Ganter B., Wille R.: *Formal Concept Analysis. Mathematical Foundations.* Springer-Verlag, Berlin, 1999.
[10] Guigues J.-L., Duquenne V.: Familles minimales d'implications informatives resultant d'un tableau de données binaires. *Math. Sci. Humaines* **95**(1986), 5–18.
[11] Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.
[12] Hájek P.: On very true. *Fuzzy sets and systems* **124**(2001), 329–333.
[13] Hájek P., Havránek T.: *Mechanizing Hypothesis Formation. Mathematical Foundations for a General Theory.* Springer, Berlin, 1978.
[14] Takeuti G., Titani S.: Globalization of intuitionistic set theory. *Annals of Pure and Applied Logic* **33**(1987), 195–211.